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DISC-O-TIC: A DISCRETE-TIME ANALYTICAL META-MODEL FOR USE IN COMBAT SYSTEMS STUDIES THAT UTILIZE HIGH RESOLUTION SIMULATION MODELS

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DISC-O-TIC: A Discrete-Time Analytical Meta-Model for Use in Combat Systems Studies that Utilize High-Resolution Simulation Models

by

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This paper provides various meta-models for extending or extrapolating in time, and varying and enhancing in coverage capability, the 2-day output of a high-resolution simulation model, here specifically but not exclusively, the Army's COSAGE. The models we propose, generically called *DISC-O-TIC*, (Discrete-Time Analytical Meta Model) are tailored to employ the discrete-time output of COSAGE and, potentially, many other such models. The model parameters are estimated from data available from a high-resolution simulation model; in the case of COSAGE the killer/victim scoreboards are used. The models are used to compute/estimate, in spread-sheet format, future force sizes and compositions that result from mutual attrition, as well as ammunition expenditures. Meta-model examination can and is shown to reveal apparent anomalies in data. Meta-models that reflect environmental variations and adaptable firing (ATCAL-like) firing rates illustrate those effects for long (8-day) battles.

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DISC-O-TIC

A Discrete-Time Analytical Meta-Model for Use in Combat Systems Studies that Utilize High-Resolution Simulation Models

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EXECUTIVE SUMMARY

This paper provides various models for extending or extrapolating in time, and varying and enhancing in coverage capability, the output of a high-resolution simulation model, here specifically but not exclusively, the Army's COSAGE. The models we propose, generically called *DISC-O-TIC*, which is short for "Discrete-Time Analytical Meta-Model) are tailored to employ the discrete-time output of COSAGE and, potentially, many other such models. The state variables of the two opponent forces are the numbers of vehicles or platforms of a variety of types for both sides, measured at regular (discrete) times only: e.g. 12-hour intervals, 1-day intervals, 2-day intervals, etc. as selected by the analyst. The outputs are survivors of the battle and munitions expended after a given number of combat days.

COSAGE output data has been made available by the Center for Army Analysis (CAA) as (i) the total forces of each side, Red and Blue, by platform or vehicle type, present at time 0 for each of several specified postures (in our specific numerical examples we analyze Blue Attack vs. Red Hasty Defense), then followed (ii) by the

corresponding forces at time 2h = 2 days = 48 hours later. Data furnished concerning engagement rates and kills by various platforms and weapons on each side allow one to compute average engagement/shooting rates and estimated kill probabilities for the various Red platforms vs. those of Blue, and *vice versa*, given the nature of the platform-weapon–target relation: the distinction between direct "aimed" fire and general "area" fire. Targets are in principle vulnerable to both.

Several new discrete-time dynamic analytical/mathematical models have been fitted to the above data by determining/estimating simple shooting and survival parameters from the above data. These simple parametric representations, so-called *meta-models,* are then used to compute/estimate, in spread-sheet format, future force sizes and compositions that result from mutual attrition, as well as the corresponding expenditures of ammunition. No explicit C4/ISR or Information Operations modeling has been carried out, but such effects as are present in COSAGE simulations are implicitly represented in the parametric representations proposed. It is then shown, for instance, that tactical-level shooting doctrine can be modified with trivial changes in kill probability parameters in the meta-models: changes from kill probabilities associated with single shots at acquired targets, to salvoes, to shoot-look-shoot, etc., can be accomplished off-line and then numerically inserted into the meta-model computations. This opportunity renders painless certain kinds of tactical what-if analyses. For more detailed discussion see Appendix A, *Firing Options and Immediate (Imperfect) BDA*

The meta-models are also capable of economically and efficiently representing likely shooting-rate re-allocation to account for target attrition. The adjustments made are speculative, or else related to that in ATCAL, an Army model analogous to those we propose. We provide some numerical illustrations of these.

We can also represent dynamic feedback adjustment of fire rates that, if overdone, could result in unstable, even chaotic, behavior.

Fitting analytical meta-models (DISC-O-TIC) and examination of the results as compared to the COSAGE data for the initial 2-day calibration period can be instructive. For example, in the current posture of Blue Attack and Red Hasty Defense the given meta-models for Blue survivorship all tend to fit the COSAGE data better than do those for Red. This is likely to be the result of far greater attrition of Red platforms than Blue (relatively few Red survivors); small numbers of remaining targets tend to induce extra shooting unless preventive steps are taken; this effect is noticeable on a suitable graph. A second example has been noticed, in which COSAGE attrition was significantly greater than that of the fitted model, is that of a Blue tracked, lightly-armored forward scouting vehicle, the UM3CFV. The considerable vulnerability of this platform stands out when a platform-by-platform plot of COSAGE vs. meta-modeled survivorship is examined.

It is strongly recommended that some form of meta-modeling be used to back up *every* complicated simulation. Taking the results of complicated simulations at face value without intensive diagnosis and critical scrutiny can be seriously misleading, particularly if later users rely on the report that the tool has previously been used by others and "therefore" need not be critically examined for a new application.

1. Introduction and Motivation

Under various circumstances it is natural to model combat attrition in *discrete time*, i.e. in equal time steps of 1-day (24 hours) or even of 2 or 3-day durations. This is because available data from higher-resolution tools, such as the Army's COSAGE, may be reported on a daily or less frequent basis, and also because realistically-sized model studies, e.g. of theater-level campaign scale may require iterative computations (optimizations, or goal programming) that are best performed on a fixed cycle schedule, where the time steps are not too small. Attrition (or suppression) calculations must fit smoothly into the overall model computation, which is often on a coarse time scale.

For example, work being carried out for J-8 to quantify ammunition consumption in various postures involves a variety of computational operations, including different attrition calculations, as time advances. Consequently it is efficient to allow initially for rather sizable time steps, on the order of several days, to keep computational effort under control.

Traditional combat/attrition modeling has overwhelmingly often been carried out in "continuous time," often using the language of differential equations; see Taylor (1983). The best known of these are the classical Lanchester equations, which have been vastly generalized (and roundly condemned, with some justification). Of course all but the simplest combat situations prohibit use of "closed form" solutions to such equations, so computational solutions have been developed; these actually are carried out by discretizing time, often on a very fine scale. Numerical studies that involve accurate solution of such equations can be quite time consuming, especially if they are only part of a much bigger effort.

Both data availability and computational burden thus urge consideration of *discrete-time* (daily, or several-day time step) model formulations for the present purpose. An additional argument for such as these is that it is easier to enhance the simplest (discrete time) models by introducing adaptive feedback from system states in the discrete-time format, and also easier to accommodate certain stochastic effects, such as inter-agent visibility, environmental and terrain influences, etc., than it is in continuous time (which models must be discretized to compute with, in any case, as pointed out above). We carry out some such enhancements in what follows, leaving others for future attention.

Finally, discrete-time models are often more easily understood by an audience uncomfortable with the ideas of calculus: derivatives and integrals.

2. Model 1.1

We work in terms of state variables that count/enumerate the numbers of Red and Blue opponents of various types and capabilities in each of several (sub)regions at times $0, h, 2h, 3h, \ldots, 37h, \ldots$. However, initially we discuss the simplest such model, one in which a single-type Blue force is in combat with a single-type Red force. A simplified deterministic "fluid approximation" presentation is given here, and made general in later sections. A stochastic version is presented in Appendix A.

2.1 Stripped-Down Illustration: Discrete-Time Direct/Aimed/Allocated Fire

Begin by assuming that Red (enemy) and Blue (own) forces confront each other in a region with unspecified dimensions (but only temporarily; generalization follows). The forces are of the same type here, e.g. are both tank forces. During equal consecutive non-overlapping time periods of duration h (e.g. single day, or consecutive several-day periods) elements or the forces search for each other and exchange shots. Assume an *aimed/assigned fire discipline* on both sides: at time t (e.g. day, 3-day step) the Blue force size $B(t)$ is effectively divided by the number of Reds, $R(t)$ (both $\{B(t)\}$ and $\{R(t)\}$ are treated here as deterministic fluids – the "mean field approximation" that can reasonably approximate the mean of a Markov chain when numbers are large). Let $B(t)/R(t)$ be the number of Blues in a specified group that are assigned to each particular Red at the beginning of period t (fractional numbers are admitted and interpretable by assuming a shooter processor-sharing strategy). Thus each Blue has his exclusive target list in period t; re-assignments of live Shooters (Blue or Red) take place each period. On average, each Red gets the attention of ~ $B(t)/R(t)$ Blues during time period $(t, t + h)$; each Red thus may experience $B(t)/R(t) \cdot n_{BR}(\bullet)h$ shots, where $n_{BR}(\bullet)h$ denotes the rate of shot-fire by each member of a Blue subgroup, the "•" notation refers to the possibility of describing the effects of other influential variables, such as the number of live Red targets: if that number becomes small then the rate of fire at it *might* also be reduced.(as in ATCAL: see Appendix D). If $n_{BR}(\bullet)$ is taken to be the (constant) maximum acquisition and shooting rate of a Blue, then the effective Blue shooting rate may potentially be unrealistically large, meaning that many Shooters (Blue) simultaneously "pile on" a possibly hapless victim (if the "victim" is already dead, or is a false target or decoy, the *Shooters* are hapless, at least as represented by the first model of this type). Of course this is a possible tactical option that is worth evaluation. It may be more munition-economical than the "pile-on" option.

A complementary setup is assumed for Reds: each Blue is assigned a specific group of $R(t)/B(t)$ Red targets at the beginning of period t; the specific groups are disjoint, as before, with $R(t)/B(t)$ assigned to each Blue. This symmetry need not be the case, but is assumed here: available data from simulations seen by these authors can support no more detail. What-if exploration using the meta-model is relatively easy.

Suppose combat occurs: compute as though each side fires simultaneously (a convenient assumption, given the basic data currently available). Postulate that a single representative kill probability prevails for Red shooter vs. Blue targets, denoted κ_{RB}, and κ_{BR} for Blue shooter against Red target. Let the corresponding one-shot *survival probabilities* be denoted by $\overline{\kappa}_{RB} = 1 - \kappa_{RB}$ and $\overline{\kappa}_{BR} = 1 - \kappa_{BR}$. These initial assumptions will be relaxed subsequently to account for other sources of variability, in general associated with the changing environment and both systematic and random/stochastic many-source effects. However, it is currently impossible to assign "hard" numbers to these effects since data are not available.

The probability that a single Red survives for one period is $\overline{\kappa}_{BR}^{n_{BR}(\bullet)hB(t)/R(t)}$; correspondingly, the probability that a single Blue survives is $\overline{\kappa}_{RB}^{n_{RB}(\bullet)hR(t)/B(t)}$.

Thus the expected number of Reds that survive a time period $(t, t+h)$ is the number at $t+h$, so these equations result

$$\boxed{R(t+h) = R(t)\overline{\kappa}_{BR}^{n_{BR}(\bullet)h\,B(t)/R(t)}} \tag{2.1}$$

$$\boxed{B(t+h) = B(t)\overline{\kappa}_{RB}^{n_{RB}(\bullet)h\,R(t)/B(t)}} \tag{2.2}$$

(again, no reinforcements or withdrawals are modeled). Given the parameters, equations (2.1) and (2.2) are easily solved recursively, e.g. on a spreadsheet, given the parameters κ_{BR} and κ_{RB}, $n_{BR}(\bullet)$ and $n_{RB}(\bullet)$ and initial forces $B(0)$ and $R(0)$.

The above *roughly* represents dynamic mutual attrition between two homogeneous sectorized or "aimed-fire" forces in a fixed "basic" time period, e.g. one day under coordinated aimed-fire conditions. It *does not attempt* to model the moment-to-moment progress of a battle, but, given daily (say) data $R(0)$, $R(h)$, $R(2h)$, $R(3h)$, …; $B(0)$, $B(h)$, … $B(13h)$, … one can deduce numerical values of the (daily or other period) average parameters κ_{BR}, κ_{RB}, $n_{BR}(\bullet)$ and $n_{RB}(\bullet)$, where the latter are treated as constants; other options are candidates for exploration, but supporting *data* is non-existent. However, with enhanced models, plausible, if speculative, doctrine may be investigated, and sensitivity testing carried out. The "data" $\{R(jh), B(jh)\}$ can come from runs of much higher resolution models (e.g. COSAGE); some results of using such to "fit" the models are reported subsequently. The "fitted" models are subsequently used to extrapolate the high-resolution results.

See Appendix A for a (*one of many possible*) stochastic version of the above model.

2.2 Connection with Classical Lanchesterian Differential Equations

Suppose that the time step duration, $h \to 0$. Expand the exponential in two Taylor terms, letting $\bar{\kappa}_{BR} = 1 - \kappa_{BR}$

$$R(t+h) = R(t)\left[1 + n_{BR}(\bullet)\ln\bar{\kappa}_{BR}\, h\frac{B(t)}{R(t)} + \mathrm{O}(h^2)\right]$$

so, subtract $R(t)$ and divide by h to find

$$\frac{R(t+h) - R(t)}{h} = n_{BR}(\bullet)\ln\bar{\kappa}_{BR}B(t) + \frac{\mathrm{O}(h^2)}{h}$$

and in the limit as h tends to zero,

$$\frac{dR(t)}{dt} = n_{BR}(\bullet)\ln\bar{\kappa}_{BR}B(t)$$

when $B(t)$. is positive; otherwise the derivative must be taken to be zero, an important boundary condition..

For small κ_{BR}, $\ln\bar{\kappa}_{BR} \approx -\kappa_{BR}$, and

$$\frac{dR(t)}{dt} = -\kappa_{BR}n_{BR}(\bullet)B(t) \tag{2.3,a}$$

and likewise

$$\frac{dB(t)}{dt} = -\kappa_{RB}n_{RB}(\bullet)R(t) \tag{2.3,b}$$

again, when the rhs is positive. The equations (2.3 a&b) are recognized as the classical "square-law" Lanchester equations. It must be recalled that unless the shooting rates $n_{BR}(\bullet)$ and $n_{RB}(\bullet)$ are explicitly forced to depend on $R(t)$ in (2.3,a) on $B(t)$ in (2.3,b), with the right-hand sides set equal to zero when $R(t) = 0$, or $B(t) = 0$, the equations are incomplete and do not respect physically appropriate boundary conditions; without imposing these constraints "solutions" can go negative or exceed initial force levels! Such conditions are *automatically* respected in (2.1) and (2.2) even if $n_{BR}(\bullet)$ and $n_{RB}(\bullet)$ are taken as (often unrealistically) constant in (2.1) and (2.2) for large $B(t)$ divided by small $R(t)$.

2.3 A Queuing Interpretation and Generalized Lanchesterian Results

It may be reasonable to conceptualize opponents as virtually "queuing up" for detection and attack by each other. For the moment, consider a continuous-time ($h \rightarrow 0$) "fluid" model of such queuing. For instance, think of $R(t)$, the surviving Reds, as being in a virtual waiting (and service) line for $B(t)$ Blue servers. Then, if one uses an analytical approximation to queuing delay effect introduced by Rider (1976), and Agnew (1976), and later discussed by Filipiak (1988),

$$\frac{dR(t)}{dt} = -\underbrace{\xi_{BR} B(t)}_{\substack{\text{maximum} \\ \text{service rate} \\ \text{(Blue)}}} \cdot \frac{R(t)}{\underbrace{1+R(t)}_{\substack{\text{queue + in-service} \\ \text{(Red)}}}} \tag{2.4,a}$$

and likewise

$$\frac{dB(t)}{dt} = -\underbrace{\xi_{RB} R(t)}_{\substack{\text{maximum} \\ \text{service rate} \\ \text{(Red)}}} \cdot \frac{B(t)}{\underbrace{1+B(t)}_{\substack{\text{queue + in-service} \\ \text{(Blue)}}}} \quad . \tag{2.4,b}$$

Such equations also arise in biomathematics, e.g. in enzyme kinetics, (also a combat situation), and may take the more general form

$$\frac{dR(t)}{dt} = -\xi_{BR} \frac{B(t) \cdot R(t)}{1+R(t)/K_R} \tag{2.5,a}$$

$$\frac{dB(t)}{dt} = -\xi_{RB} \frac{R(t) \cdot B(t)}{1+B(t)/K_B} \tag{2.5,b}$$

where K_R and K_B are *Michaelis-Menten* constants; see Murray (1989). Their presence allows further opportunity for fitting to experimental (simulation) data.

The virtue of this type of formulation is that it gracefully bridges the gap between a target-rich environment, e.g. for B as in (2.4,a), when R is *large*:

$$\frac{dR(t)}{dt} \cong -\xi_{BR} B(t) \tag{2.6}$$

which yields the Lanchester Square Law of (2.3) above; if R is *small*, so Blue is target-poor, i.e. Red targets are much less easily found and killed, we find (roughly)

$$\frac{dR(t)}{dt} \cong -\xi^{\#}_{BR} B(t) R(t) \tag{2.7}$$

which is the Lanchester Linear Law; here the appearance of the product, $B(t)R(t)$, represents the random contact rate between opposing force elements. Owing to the smooth saturation of the shooter, the search-kill-rate $\xi^{\#}_{BR} < \xi_{BR}$ in the former equation is smaller because more time is spent searching for and identifying targets before shooting. Of course the same behavior may occur for Blue; but there may well be asymmetry between force behaviors.

Now go in the time-honored Lanchester direction. Division of, say, (2.4,a) by (2.4,b) gives

$$\frac{dR}{dB} = \frac{\xi_{BR}}{\xi_{RB}} \cdot \frac{(1+B)}{(1+R)} \tag{2.8}$$

which, upon integrating, yields

$$\boxed{\frac{R^2(t)}{2} + R(t) - \frac{R^2(0)}{2} - R(0) = \frac{\xi_{BR}}{\xi_{RB}} \left(\frac{B^2(t)}{2} + B(t) - \frac{B^2(0)}{2} - B(0) \right)} \tag{2.9}$$

a naturally blended version of the Square and Linear laws. The Michaelis-Menten generalization of (2.5) is seen to produce

$$\boxed{\frac{R^2(t)}{2K_R}+R(t)-\frac{R^2(0)}{2K_R}-R(0)=\frac{\xi_{BR}}{\xi_{RB}}\left(\frac{B^2(t)}{2K_B}+B(t)-\frac{B^2(0)}{2K_B}+B(0)\right)} \qquad (2.10)$$

which allows further tuning between the "pure" laws. Generalizations are possible, see Gaver and Jacobs (1997), but are omitted for the present. Taylor's books (1980, 1983) are basic sources for this topic.

2.4 Extra/Over-Variability in Model Response

The previous model types are not now, *but can be made*, explicitly responsive to either systematic and explanatory/regression variables, or to random ("hidden") sources of variability within a scenario or posture. Among such system effectors can be range and visibility conditions between Shooter and Target, and maneuvers taken to change these. *One* approach to include these effects is to introduce a (one or more) parsimonious parametric modifications of the survivor probability, $\overline{K}$; the latter may be patterned on the medical-biological survivor-probability methodology, Cox and Oakes (1984), that incorporates a *hazard* function, deterministic and/or random. This leads to introduction of a revised survival probability

$$\overline{K}(t)=e^{-\boldsymbol{h}(t;x(t);\varepsilon(t))} \qquad (2.11)$$

here $\boldsymbol{h}(t; x, \varepsilon(t))$ is a (partially) *random hazard rate*. In classical military language this resembles (but is complementary to and generalizes) a *Killer-Victim Scoreboard*; it is a generalization that includes systematic observable explanatory variables, plus over-variability from many causes.

<u>Example 1</u>: Range/Distance Effects Explicitly Modeled

The data available to parameterize the type of model discussed, e.g. COSAGE output, may well

(a) record $d_{BR}(t)$, a *characteristic range* between (segments of) the two forces (here considered) at time $t = kh$. Other things being equal (they will not be!) we anticipate that a hazard component due to range alone can have the form

$$\overline{\kappa}_{BR}(t) = \overline{\kappa}_{BR} f(d_{BR}) \tag{2.12}$$

where the basic constant $\overline{\kappa}_{BR}$ and the function, $f(\bullet)$, of distance are parameters to be determined. This form allows the observed survival probability to be exactly matched at two specific ranges, and be adjustably decreasing with increasing range, as $d_{BR}(t)$ increases. Of course other explanatory variables such as terrain cover and its usage also influence kill probability. Unfortunately, the information available from COSAGE at present kill prevent explicit representation of such effects.

<u>Example 2</u>: Random Period-to-Period Variation

Think of $R(t, \varepsilon(t))$ as the expected or mean value of Red force size, *conditional on* $\varepsilon(t)$, a random environmental effect that persists throughout the t^{th} period; for the moment take $\{\varepsilon(t), t = h, 2h, \ldots, \}$ to be *independently* and *identically* distributed (iid); however, the analytical/mathematical difficulties do not increase at all if the distributions of the environmental effects are not identical, e.g. depend on time but remain independent. *If* the $\varepsilon(t)$s are common to $B(t)$ and $R(t)$ throughout, as could represent common weather conditions, then we can put, conditionally,

$$\begin{aligned} R(h,\varepsilon(1)) &= R(0)\overline{\kappa}_{BR}^{\varepsilon(1)n_{BR}(\bullet)hB(0)/R(0)} \\ B(h,\varepsilon(1)) &= B(0)\overline{\kappa}_{RB}^{\varepsilon(1)n_{BR}(\bullet)hB(0)/R(0)} \end{aligned} \tag{2.13}$$

Remove the condition on $\varepsilon(1)$ to obtain

$$\overline{R}(h) = R(0)E_{\varepsilon}\left[e^{\varepsilon(1)n_{BR}(\bullet)h(B(0)/R(0))\ln\overline{\kappa}_{BR}}\right] \equiv R(0)\Psi\left[n_{BR}(\bullet)h(B(0)/R(0))\ln\overline{\kappa}_{BR}\right]$$
$$\overline{B}(h) = B(0)E_{\varepsilon}\left[e^{\varepsilon(1)n_{RB}(\bullet)h(R(0)/B(0))\ln\overline{\kappa}_{RB}}\right] \equiv B(0)\Psi\left[n_{RB}(\bullet)h(R(0)/B(0))\ln\overline{\kappa}_{RB}\right] \tag{2.14}$$

where $\Psi(\bullet)$ is the Laplace transform of $\varepsilon(t)$,

$$\Psi(s) = E_{\varepsilon}\left[e^{-s\varepsilon(t)}\right]$$

and where, in the present application,

$$s = -n_{BR}(\bullet)h(\overline{B}(t)/\overline{R}(t))\ln\overline{\kappa}_{BR}$$
$$> 0 \qquad \text{since} \quad \ln\overline{\kappa}_{BR} < 0.$$

On the fluid-approximation principle, see Appendix A, calculate

$$\overline{R}(t+h) = \overline{R}(t)E_{\varepsilon}\left[e^{\varepsilon(t+h)n_{BR}(\bullet)h(\overline{B}(t)/\overline{R}(t))\ln\overline{\kappa}_{BR}}\right] \equiv \overline{R}(t)\Psi\left[n_{BR}(\bullet)h(\overline{B}(t)/\overline{R}(t))\ln\overline{\kappa}_{BR}\right]$$
$$\overline{B}(t+h) = \overline{B}(t)E_{\varepsilon}\left[e^{\varepsilon(t+h)n_{RB}(\bullet)h(\overline{R}(t)/\overline{B}(t))\ln\overline{\kappa}_{RB}}\right] \equiv \overline{B}(t)\Psi\left[n_{RB}(\bullet)h(\overline{R}(t)/\overline{B}(t))\ln\overline{\kappa}_{RB}\right] \tag{2.15}$$

Note that the above only focuses on the *mean* survivorship of each side. In fact, the commonality of the $\varepsilon(t)$-effect induces positive association or "correlation" between the attrition rates.Operational implications of this phenomenon are not investigated for the present.

Illustration: Let $\{\varepsilon(t)\}$ be iid and Gamma, with mean 1 and shape parameter β, $0 < \beta < \infty$, (variance $1/\beta$). Then since

$$\Psi(s) = \left[\frac{1}{1+s/\beta}\right]^{\beta} \tag{2.16}$$

we obtain

$$\overline{R}(t+h) = \overline{R}(t)\left[1 - \frac{n_{BR}(\bullet)h}{\beta}\left(\overline{R}(t)/\overline{B}(t)\ln\overline{\kappa}_{BR}\right)\right]^{-\beta} \tag{2.17}$$

$$\bar{B}(t+h)=\bar{B}(t)\left[1-\frac{n_{RB}(\bullet)h}{\beta}\left(\bar{R}(t)/\bar{B}(t)\ln\bar{\kappa}_{RB}\right)\right]^{-\beta}. \tag{2.18}$$

Numerical results are easily obtained using spreadsheet techniques. Statistical analysis of data to support this model, i.e. to estimate the value of beta, has not yet been undertaken. However, the model can be used as a what-if tool to study the numerical effect of a plausible model change on survivability and munitions expenditure.

The dependence of the (mean) survivorship of Red and Blue is seen to be systematically affected by the *dispersion* of the $\varepsilon(t)$-distribution. The mean of that distribution has been fixed at unity. If β becomes large ($\beta \to \infty$) the variance ($1/\beta$) hence standard deviation $\left(1/\sqrt{\beta}\right)$ of $\varepsilon(t)$ becomes negligible, and the original model reappears.

If β is small, i.e. close to or below unity, the survivorship shape changes, and radically if $\beta < 1$: relatively high survivorship occurs for small effective values of killing rate $n_{BR}(\bullet)\ln\bar{\kappa}_{BR}$, with relatively slower trail-off as $-n_{BR}(\bullet)\ln\bar{\kappa}_{BR}$ increases.

The above model can be generalized in various ways. For example, the $\varepsilon(t)$-effect applied to one side for several specified periods *can* represent the use of an obscurant or gas (necessitating use of movement-inhibiting protection) during several days of the campaign. This certainly tends to reduce the capability of the Shooters on the receiving side of the obscurant. The effectiveness can be quantified in the present simplified model, but is worth doing only if a realistic variety of weapon types is in operation. This more realistic situation is a candidate for consideration in later work. .

3. Model 2: Multitype Shooters and Targets

We work in terms of state variables that count/enumerate the numbers of Red and Blue opponents of various types and capabilities at times 0, h, $2h$, $3h$, …, $37h$, …

Note: high-resolution data *by location in a (sub)region* are often not reported *as output* in existing simulation models (such as COSAGE), although the model itself evolves spatially. *Analytical opportunities are lost by hiding the spatial-temporal aspect of* *combat from the COSAGE user. It is recommended that more such detail be made available to the user/analyst.*

3.1 State Variables

$R(j_R; t)$ = Number of Red platforms of type j_R in region i_R at time t

$B(j_B; t)$ = Number of Blue platforms of type j_B in region i_B at time t

3.2 Parameters

$\rho_B(j_B, w_B; t)$ = rate of fire of weapon type w_B from Blue platform type j_B in region i_B during $(t, t+h]$

$\omega_{BR}(j_B, w_B; j_R; t)$ = fraction of weapons of type w_B from platform of type j_B fired at Red platforms of type j_R during the time interval $[t, t+h]$.

$\kappa_{BR}(j_B, w_B; j_R)$ = probability of kill for Blue platform of type j_B firing with weapon w_B at Red platform of type j_R

Note: Attrition need not be permanent ("kill"): by expanding the state space we accommodate partial (e.g. mobility) kills, and perhaps temporary psychological kills ("suppression"); however, recovery from suppression is not explicitly modeled here, but to do so is not difficult. Again, data support for this is not available explicitly from current COSAGE data.

3.3 Aimed or Direct Fire

This model type is useful for representing the attrition of Red by Blue (and vice versa) when Blue weapons can be allocated to exercise "aimed" or "direct" fire at batches of Reds whose type and location is presumed known (visible either directly or indirectly) to Blue shooters. Tank warfare is an example. The total shots capable of being fired *at* Red targets of type j_R *by* Blue platforms of type j_B firing weapon w_B in $(t, t+h]$ is

$$\begin{aligned} N_{BR}(j_B, w_B; j_R; t) &= B(j_B;t)\rho_B(j_B, w_B;t)\omega_{BR}(j_B, w_B; j_R; t)h \\ &\equiv B(j_B;t)\left[n_{BR}(j_B, w_B; j_R; t)h\right] \end{aligned} \tag{3.1}$$

In words, $n_{BR}(\bullet)h$ is the number of shots capable of being fired by an individual type j_B Blue platform at t against Reds of type j_R during $(t, t+h]$. The number of Reds of type j_R at time $t+h$ (no reinforcements) is obtained by the argument of Section 2 preceding: the basic notion of aimed or direct fire is, here, that a certain number of the total shots fired by a platform in a given category is allocated to each single Red unit, type j_R. Only a subset of Blues is allocated to each of a particular subset of Red (Blues do not target totally at random, nor do Reds). In the present deterministic/fluid-like approximation this suggests that each one such Red is the target of a potential $N_B(j_B, w_B; j_R; t)/R(j_R; t)$ shots. The probability that such a target survives for one h-period is

$$\begin{aligned} S(j_R;t) &= \prod_{j_B}\prod_{w_B}\left(1-\kappa_{BR}(j_B,w_B;j_R)\right)^{N_{BR}(j_B,w_B;j_R;t)/R(j_R;t)} \\ &\equiv \prod_{j_B}\prod_{w_B}\overline{\kappa}_{BR}(j_B,w_B;j_R)^{B(j_B;t)n_{BR}(\bullet,j_B,w_B;j_R;t)h/R(j_R;t)} \end{aligned} \tag{3.2}$$

from which come the survival recurrences (assuming both sides behave in the same way).

Model 2.1:

$$\boxed{\begin{aligned} B(j_B;t+h) &= B(j_B;t)S(j_B;t) \qquad (3.3,\text{a}) \\ R(j_R;t+h) &= R(j_R;t)S(j_R;t) \qquad (3.3,\text{b}) \end{aligned}}$$

where the probability of a Red of type j_R surviving aimed fire (AF) for one time unit, h, is thus given by the right-hand side of (3.2). An equivalent formula holds for Red DF vs. Blue. We now abbreviate the number of shots per Blue at a Red j_R as $n_B(\bullet, j_B, w_B; j_R; t)$. As before, in Section 2, we use the "•" to indicate an unspecified (to-be-specified, but otherwise taken to be constant, usually representative of the first few days of COSAGE data) dependence of the number of shots on either deterministic or stochastic sources of variation.

3.4 Firing Rate Allocation

As remarked in Section 2, it is not satisfactory or realistic to let a firing rate, e.g. of Blue weapon w_B from a platform type j_B, against a Red of type j_R, i.e.

$$n_{BR}\left(j_B, w_B; j_R, t\right)$$

remain *constant* over time, as opponent numbers change (*decrease only* in this model): if $B(j_B; t)$ becomes relatively large compared to $R(j_R; t)$ then an unrealistic number of shots may be fired at a Red target — one that in practice is either already dead, or may have moved!

The numbers of shots of various types fired at a perceived-alive target are actually decision variables. The decision is guided by rules of doctrine, and training, but must rely in real life on a tank commander's individual skill and perceptiveness. Our present models do not directly account for variability of operator performance. In the context of the current model type, and others more detailed and high-resolution, we will suggest and test some rules that behave with qualitatively correct properties. For the first model we simply take firing rates as constants, estimated from data (COSAGE). This assumption is subsequently modified.

Some Shot Allocation Rules

The "historical" COSAGE run record, of 2h=48 hours' duration for this report's sample data set has been analyzed to develop estimates of shooting rates and kill probabilities for platform-weapon combinations against specified target types.

As noted, in the first model the above rates and probabilities are not altered (they remain constant) over time: the same shooting rates per specified Blue Shooter vs. Red target, and *vice versa* are assumed to prevail throughout: these are the Shooter vs. Target rates that characterize the initial 48 hours of the COSAGE simulation data. It is justifiable to question this assumption, since presumably total rate of fire directed at individual targets will diminish as the number of targets per assigned shooters decreases. This may occur because of increased difficulty of acquiring alive eligible targets.

Here are some rules:

(a) Initial (0,2h] = 48 hour shooting rates per shooter maintained throughout the elapsed battle time.

(b) Initial shooting rates redistributed each time period across alive targets, in same proportion as during initial period (0, 2h].

(c) Initial shooting rates redistributed each time period in proportion to number of surviving targets in each target class. The redistribution need not be *proportional* but can be some general or weighted function that is selected for control purposes.

(d) Initial shooting rates redistributed according to the perceived lethality (against appropriate associated shooter platforms) of each target class. Approximately and myopically: weight the surviving numbers of each target class by the probability of kill, or rate of kill, against its shooter-class targets. In short, a shooter of either side may well tend to shoot first at its currently most dangerous opponent.

(e) Initial *gross* shooting rates maintained constant in each subsequent shooter-target pairing. This is equivalent to applying the logic of the ATCAL model assumptions in each time period; see Chap. 6, expression (6.2.1) of Caldwell, *et al.* (2000). In turn, this amounts to adjusting the engagement-killing rate so as to keep the target type attrition rate constant during each day of the battle, which leads to an exponential decrease in targets throughout the battle.

Comments: There appears to be no basic reason why such a rule is especially natural or "optimal". It does appear to be part of one standard Army analytical tool, namely ATCAL, and hence is worth examination in the present context. It implies a certain shooter restraint: if the number of shooters vs. members of a class of alive targets becomes greater than was the case initially, then the shooters' shooting rate is reduced so that the original shooting-killing rate prevails in the new time interval. This may be qualitatively plausible in some rough sense. Why the adjustment to achieve *exponential* decrease of target forces might be descriptive of real combat remains mysterious.

The basic direct-fire allocation of Blue platforms to Red targets in the present model begins by assigning a set of Blue platforms exclusively to a particular set of Reds. Consequently, not every Blue platform in the situation represented can acquire or shoot at *any* Red platform; there may be different assignments throughout a basic period of duration h (e.g. a day), but it is presently assumed that those assignments are respected. Given the data available, it is the basic current assumption that, during period $(t, t+h)$ each Red of type j_R is a candidate to experience shots from some Blues of type j_B: e.g. during period t (e.g. day t to t+h) each Red of type j_R receives on average Blue shots of weapon type w_B from platforms of type j_B that number

$$\begin{matrix}\text{Blue } w_B \text{ shots at a Red of type } j_R \\ \text{during time period } (t, t+h)\end{matrix} = \frac{\sum_{j_B} n(\bullet, j_B, w_B; j_R; t) B(j_B; t)}{R(j_R; t)}. \tag{3.4}$$

Data available from COSAGE runs provides the possibility of calculating/estimating the *initial* shooting rate of a Blue type j_B against a Red target, type j_R:

$$\hat{n}(\bullet, j_B, w_B, j_R; h) = \frac{\#\text{ of } w_B\text{ - shots by all } B(j_B, 0) \text{ in } (0, h)}{B(j_B, 0)}. \tag{3.5}$$

The rate of shooting is very likely to be strongly dominated by the rate of target acquisition: the time between shots is mainly the time between successful detection, acquisition, and classification (or misclassification) of a target; the actual shooting time (weapon firing and delivery on target) is negligible by comparison.

3.5 Poisson Shooting-Rate Model: Model 2.2

Invocation of the Poisson process, Feller (1968), provides a plausible and useful next-stage model that incorporates stochastic/chance variability into target acquisition and shooting. During a generic period $(t, t+h)$ a group of Shooter platforms allocated to fire specified weapons at a specified target type search for their quarry in such a way that the number they find is random (realistically), and is governed by a Poisson random process. If the Shooter is of type j_S and the quarry of type j_Q (here if $Q = B$, $S = R$, while otherwise $Q = R$, $S = B$) the random number found, per Shooter of type j_S in $(t, t+h)$ has the Poisson distribution with rate $\lambda_{SQ}(\bullet, j_S, j_Q)$; if the Poisson random version of $n_{SQ}(\bullet, j_S, j_Q)h$ has mean $\lambda_{SQ}(\bullet, j_S, j_Q)h$ then it can be shown that (3.3) is replaced by (3.6,b) below, with

$$B(j_B, t+h) = B(j_B, t)e^{-\sum_{j_R}\sum_{w_R}\lambda_{RB}(\bullet, j_R, w_R; j_B)h\,\kappa_{RB}(j_R, w_R; j_B)\left(R(j_R, t)/B(j_B, t)\right)} \tag{3.6,a}$$

and

$$R(j_R, t+h) = R(j_R, t)e^{-\sum_{j_B}\sum_{w_B}\lambda_{BR}(\bullet, j_B, w_B; j_R)h\,\kappa_{BR}(j_B, w_B; j_R)\left(B(j_B, t)/R(j_R, t)\right)} . \tag{3.6,b}$$

Here we have incorporated the fraction of different weapons shot by the same platforms directly into the parameterized search rate. As before, the symbol "•" denotes the possible dependence of the acquisition latency on other explanatory variables, such as location, range, previous acquisitions experienced, etc. These variables are not included here because they cannot be obtained from current COSAGE data.

4. Connection with COSAGE Data

Note that the raw output of COSAGE data *currently made available* aggregates Red (and Blue) force types over their various locations.

COSAGE data from actual runs, or other historical data, may give numerical values for the fraction, ω_{BR} of weapons of type w_B historically fired by Blues, type j_B, vs. Reds, type j_R, and vice versa. The data allows empirical determination of the systems' attrition power.

An alternative is to (eventually) *simulate rule-based adaptivity*: the rate of fire at a given batch of Red targets, e.g. tanks at a certain location, may be made much higher than at APCs at another location; this can be achieved by adjusting the functional form of $\omega_{BR}(j_B, w_B; j_R)$; the latter can, and should, depend on (currently unmodeled) availability of ammunition in various categories, and of course the presumed threat of the different Red target batches.

4.1 Adapting the Model to *Easily* Utilize COSAGE Data

Models used to estimate the expected number killed in direct fire using parameter estimates from COSAGE are summarized below.

<u>Notation</u>

$R(j_R, t)$ = Number of Red shooters of type j_R at time t

$B(j_B, t)$ = Number of Blue targets of type j_B at time t

$N_{RB}(j_R, w_R; j_B, t)$ = (Mean) number of shots fired by all Red shooters of type j_R using munition w_R at all Blue targets of type j_B during period $(0,t]$

$N_{BR}(j_B, w_B; j_R, t)$ = (Mean) number of shots fired by all Blue shooters of type j_B using munition w_B at all Red targets of type j_R during period t: $(0,t]$

$2h$ = number of days of combat represented in the COSAGE run

r = the number of COSAGE replications; often 16.

$n_{RB}(j_R, w_R; j_B, t) = N_{RB}(j_R, w_R; j_B, 2h)/R(j_R; 0)2h = \lambda_{RB}(j_R, w_R; j_B)$

$n_{BR}(j_B, w_B; j_R, t) = N_{BR}(j_B, w_B; j_R, 2h)/B(j_B; 0)2h = \lambda_{BR}(j_B, w_B; j_R)$

$K_{RB}(j_R, w_R; j_B, t)$ = (Mean) number of Blue targets of type j_B killed by shots fired by all Red shooters of type j_R using munition w_R during $(0,t]$

$K_{BR}(j_B, w_B; j_R, t)$ = (Mean) number of Red targets of type j_R killed by shots fired by all Blue shooters of type j_B using munition w_B during $(0,t]$

$$\hat{\kappa}_{RB}(j_R, w_R; j_B) = \frac{K_{RB}(j_R, w_R; j_B, 2h) + \frac{1}{r}}{N_{RB}(j_R, w_R; j_B, 2h) + \frac{2}{r}} = \tag{4.1}$$

Bayesian estimated probability of kill of Blue type j_B by Red platform, type j_R using weapon type w_R. Uses uniform (0,1) prior for the probability of kill

$$\hat{\kappa}_{BR}(j_B, w_B; j_R) = \frac{K_{BR}(j_B, w_B; j_R, 2h) + \frac{1}{r}}{N_{BR}(j_B, w_B; j_R, 2h) + \frac{2}{r}} = \quad (4.2)$$

Bayesian estimated probability of kill of Red type j_R by Blue platform, type j_B using weapon type w_B. Uses uniform (0,1) prior for the probability of kill

First model: Model 2.1

Rates of fire observed during $(0,2h]$ in COSAGE are used for all future times. The same for kill probabilities. Both are estimated as described under Notation above.

$$B(j_B, t+h) = B(j_B, t)\prod_{j_R}\left(1-\kappa_{BR}(j_R, w_R; j_B)\right)^{n_{RB}(j_R, w_R; j_B)R(j_R, t)h/B(j_B, t)} \quad (4.3)$$

Poisson model: Model 2.2

$$B(j_B, t+h) = B(j_B, t)\prod_{j_R}\exp\left\{-\lambda_{RB}(j_R, w_R; j_B)h\kappa_{RB}(j_R, w_R; j_B)R(j_R, t)/B(j_B, t)\right\} \quad (4.4)$$

Expected number kills model: Model 2.3

$$B(j_B, t+h) = B(j_B, t) - \sum_{j_R} n_{RB}(j_R, w_R; j_B)R(j_R, t)h\kappa_{RB}(j_R, w_R; j_B) \quad (4.5)$$

if $B(j_B, t+h) > 0$ and 0 otherwise.

Fluid queuing model: Model 2.4

$$B(j_B, t+h) = B(j_B, t) - \sum_{j_R} n_{RB}(j_R, w_R; j_B)R(j_R, t)\frac{B(j_B, t)}{1+B(j_B, t)}h\kappa_{RB}(j_R, w_R; j_B) \quad (4.6)$$

if $B(j_B, t+h) > 0$ and 0 otherwise.

Random Environment Gamma model: Model 2.5

$$B(j_B, t+h) = B(j_B, t)\prod_{j_R}\left[1 - \frac{n_{RB}(j_R, w_R; j_B)\ln\left(1-\kappa_{RB}(j_R, w_R; j_B)\right)R(j_R, t)h/B(j_B, t)}{\beta}\right]^{-\beta} \quad (4.7)$$

Figures/graphs to be presented graphically present the *COSAGE data plotted vs. the fitted models*. These are results comparing the average number of LIVE platforms surviving from COSAGE and projected by the above models for each platform. Such graphs are useful diagnostic tools that can reveal anomalies in data (and point out programming errors!). COSAGE summary output reports summary statistics averaged over a specified number of replications. For aimed fire weapons the statistics include the average number of weapons of type w fired by platform of type j at a target of type k and the average number of targets of type k killed by weapons of type w fired by platforms of type j for a 2-day period. Also specified are the initial numbers of Red and Blue platforms, the number of replications and the combat posture. The kill probability of a single aimed fire weapon w_B fired by a Blue platform of type j_B against a j_R Red in situation s can be estimated from the COSAGE output as follows.

Let $\overline{K}_{BR}(j_B, w_{B;} j_R; s)$ be the average (e.g. over 16 replications) number of Reds of type j_R killed by aimed fire weapon w_B fired by Blue platforms of type j_B for the two-day period during posture s. Let $\overline{N}_{BR}(j_B, w_B; j_R; s)$ be the average number of aimed fire weapons w_B fired by Blue platforms of type j_B at Reds of type j_R for the two-day period during posture s. In aimed fire each weapon is assumed to be able to affect exactly 1 Red target; we ignore weapons fired at already killed targets. An estimate of the kill probability of a single aimed fire weapon, type w_B, shot by a Blue of type j_B against a j_R Red in posture s during the two-day COSAGE run is

$$\kappa_{BR}(j_B, w_B; j_R; s) = \frac{\overline{K}_{BR}(j_B, w_B; j_R; s)}{\overline{N}_{BR}(j_B, w_B; j_R; s)}. \tag{4.8}$$

(Note: this is actually the maximum likelihood estimate of a binomial model's kill probability, here $\kappa_{BR}(j_B, w_B; j_R; s)$. It is scenario-specific, and range or other condition dependence is not specified.)

Now to avoid having to estimate a kill probability as 0 owing to small-sample bad luck we replace $\overline{K}_{BR}/\overline{N}_{BR}$ by a Bayes estimate that assumes kills are binomial and applies a *uniform* prior (one can use a more informative prior if one is available); the result is the estimate of kill for Red of type j_R when fired upon directly by Blue of type j_B using weapon type w_B during phase s; if r is the number of COSAGE replications (e.g. 16); the Bayes probability of kill for a single shot is

$$\hat{\kappa}_{BR}(j_B, w_B; j_R; s) = \frac{\overline{K}_{BR}(j_B, w_B; j_R; s) + \frac{1}{r}}{\overline{N}_{BR}(j_B, w_B; j_R; s) + \frac{2}{r}}. \tag{4.9}$$

Thus, the probability a particular Red survives a shot is

$$\hat{\bar{\kappa}}_{BR}(j_B, w_B; j_R; s) = \left[1 - \frac{\overline{K}_{BR}(j_B, w_B; j_R; s) + \frac{1}{r}}{\overline{N}_{BR}(j_B, w_B; j_R; s) + \frac{2}{r}}\right] \tag{4.10}$$

4.2 Area or Unaimed Fire: Model 3.1

Suppose certain weapons fire in an unaimed area-fire manner. Then (4.3) must be modified to reflect the way that any shot places any Red target, type j_R, in location i_R at risk. One formula will handle this for each phase/stage s; put

$$S(j_R; t) = \prod_{j_B} \prod_{w_B} \left(1 - \kappa_{BR}(j_B, w_B; j_R)\right)^{N_{BR}(\cdot)/R(\cdot)^{\theta(w_B)}} \tag{4.11}$$

where

$$\theta(w_B) = \begin{cases} 1 & \text{if weapon } B \text{ is aimed;} \\ 0 & \text{if weapon } B \text{ is area.} \end{cases} \tag{4.12}$$

and N_{BR} is defined in (3.1). Of course the corresponding kill probabilities must be made weapon-specific. It is even possible to let the indicator $\theta(\cdot)$ depend on other features of combat, such as distance, general terrain features, visibility, etc., *and* to let $\theta(\cdot)$ take on values other than 0 or 1. *There is currently no information quoted from COSAGE runs to permit such refinements. This represents a lost opportunity for learning.*

4.3 Using Available COSAGE Data to Estimate Survival for Area Fire Weapons

In this section we provide formulas for the effect of indirect or area fire analogous to (3.3,a), (3.3,b), (3.6,a), and (3.6,b) for aimed fire. We use only data now available from COSAGE runs, so the footprint-that-affects-several-targets effect must be inferred indirectly. No doubt the process used here can be improved, but the method necessarily uses data presently available from a detailed "realistic" model, COSAGE, to "fit" a much cruder model: simple recurrences (3.3,a) and (3.3,b).

The argument below applies to a generic area-fire situation; it is easily made specific.

There are R Red targets, all of the same type (e.g. tanks), and all equally vulnerable to Blue weapons. Let there be S_B Blue indirect-fire weapon shots of specified type aimed into the region where the Reds reside. Let α be the probability that a Red (target) is affected (e.g. *susceptible* or *exposed* to being killed, but possibly damaged or suppressed within a footprint) by a Blue indirect weapon shot. Let the indicator function $\boldsymbol{I}_j(i) = 1$ if the i^{th} shot *affects* the target number j; otherwise $\boldsymbol{I}_j(i) = 0$. Then the total number of targets affected by indirect weapons during a replication of COSAGE is the sum

$$\boldsymbol{N}_f = \sum_{i=1}^{S_B} \sum_{j=1}^{R} \boldsymbol{I}_j(i), \tag{4.13}$$

a random variable, the observed value of which on replication k is $n_f(k)$, $k = 1,\ldots, r$. Now assuming symmetry over the area fired upon, $E[\boldsymbol{I}_j(i)] = \alpha$ for each shot and target (possibly more plausible if all shots are fired simultaneously so that targets do not move). Thus

$$E[\boldsymbol{N}_f] = S_B R\alpha \tag{4.14}$$

and so we can estimate α by moments:

$$\overline{n}_f = \frac{1}{r}\sum_k n_f(k); \tag{4.15}$$

consequently an estimate of α is $\hat{\alpha} = \bar{n}_f / S_B R$; the number of shots S_B is estimated by the average of the number of shots observed during the r replications of 48 hours of COSAGE combat.

Suppose ξ is the probability that an affected (within footprint) target is killed. The probability that a particular target is killed is thus $\alpha\xi$ (first affected, then killed); the probability of surviving one shot is the $1 - \alpha\xi$ for any target (symmetry). *Assume* now that the probability that a single target survives (all) S shots is $(1 - \alpha\xi)^S$; this is perhaps more plausible if the target cannot, or does not, move. In that respect it is a lower bound on a particular target's survival. Put $\kappa_w(a) = \alpha\xi$.

COSAGE records the average number of targets killed by the indirect fire weapons over the r (often $r = 16$) replications: $\bar{k}_R(2)$ for Red, while $R(0)$ is the initial number of Reds. By symmetry and the method of moments put

$$\bar{k}_R(2) = R(0)\bar{S}_B \kappa_w(a) \tag{4.16}$$

from which the probability that any Red is killed by a single Blue shot is estimated as

$$\kappa_w(a) = \frac{\bar{k}_R(2)}{R(0)\bar{S}_B} \tag{4.17}$$

where $\bar{S}_B$ is the average number of indirect-fire (Blue) weapons shot during the r replications.

To avoid having to estimate a survival probability as 1 owing to small-sample luck we replace $k_R(2)/\bar{S}_B R(0)$ by a Bayes estimate that assumes kills are binomial and applies a uniform prior; the result is the estimate of survival for Red when fired upon indirectly by Blue, using weapon type w_B: if r is the number of replications (e.g. 16),

$$\hat{\bar{\kappa}}_{BR}(w_B, j_R) = \left(1 - \frac{\bar{k}_R(2) + 1/r}{R(0)\bar{S}_B + 2/r}\right) \tag{4.18}$$

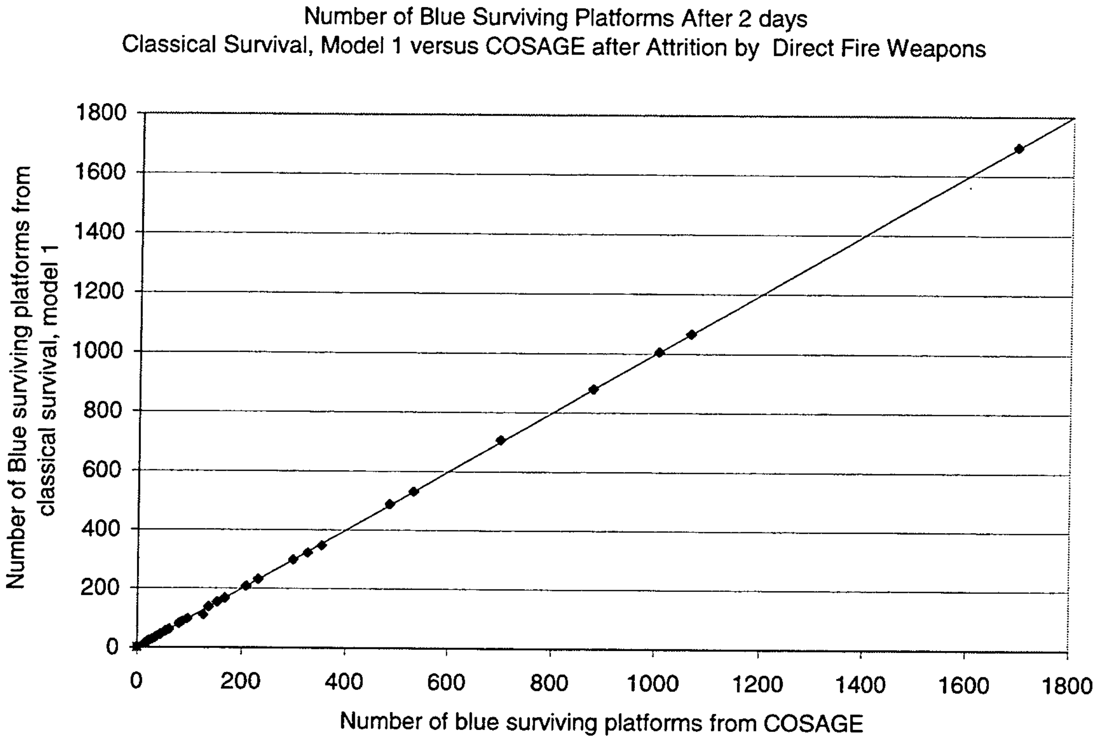
where $\overline{S}_B = B(0)n_{BR}(\cdot)$ since the above applies to Blue shooting at Red. A symmetrical formula holds when Red is indirectly shooting at Blue.

5. Numerical Illustrations

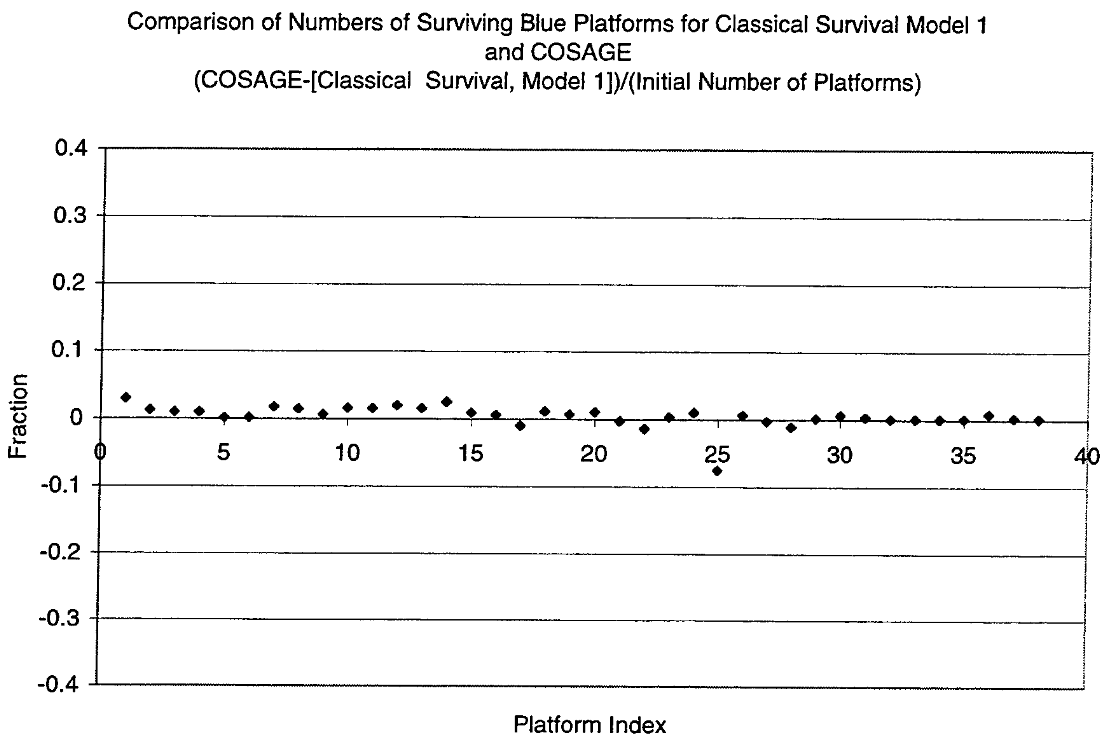
We provide a number of graphical displays of COSAGE-data supported survivorship and Munitions expenditures by various platforms. Fitted meta-models are compared to COSAGE data for the two-day time period for which COSAGE data is available for the posture *Blue attack-Red hasty defense.* Then comparisons between model types are displayed. The model types considered are the Classical Survival Model 1 (4.3), the Poisson Acquisition model (4.4) and an ATCAL-like attrition model (D.4) and (D.8). Comparison of Figures A.2 and B.2 show that the Classical Survival Model 1 summarizes COSAGE number of platforms surviving results for the Blue forces better than for the Red Forces. In this scenario, the Red forces suffer greater attrition than the Blue forces. Classical Survival Model 1 tends to underestimate the number of surviving platforms. The Blue platform with the largest negative discrepancy in Figure A.2 is Platform 25 which is the UM3CFV, a lightly armored tracked forward Blue scouting vehicle that suffers greater attrition in the Classical Survival Model 1 than in COSAGE. Figures C.1 and C.2 display the amount of Blue munitions used and the number of Red surviving platforms after 8 days for the Poisson acquisition model versus the classical survival model. The Poisson acquisition model tends to predict more munitions used for fewer Reds killed. Figures D.1 and D.2 compare the amount of Blue munitions used during 8 days for classical survival model 1 and the Poisson acquisition model with and without proportional reallocation of fires as specified in equation (E.1) of Appendix E. Not surprisingly, the re-allocation of fire results in more Blue munitions being used. However, the effect is greater for Classical survival model 1 than for the Poisson acquisition model. Figures E.1 and E.2 compare the number of surviving Red platforms after 8 days for the classical survival model 1 and the Poisson acquisition model. As expected the weapon reallocation results in a few more Red platforms being attrited. Once again, the effect is more extreme for the classical survival model 1. Figures F.1 and F.2 display the number of Blue munitions fired during 8 days and the number of Red platforms surviving after 8 days for the classical survival model 1 and the ATCAL-like model described in Appendix D. The ATCAL-like model tends to predict fewer Blue munitions expended during the 8 days than the Classical Survival model 1. The ATCAL-like model tends to predict fewer Red platforms surviving for those Red platforms that have a relatively large number surviving in Classical Survival model 1. If Classical Survival Model 1 predicts a small number of Red platforms surviving, then the ATCAL-like model tends to predict a somewhat greater number of platforms surviving.

As they are currently fit to the particular COSAGE-model posture, none of the meta-models differ enormously in their implications. This can change, for example, if the meta-models are numerically parameterized differently, i.e. if the Random Environment model (4.7) is used with the value β small, or if the ATCAL tuning parameter f is altered. The models require exploration that shows which have the most conservative but realistic assessment of kill rate for munitions expended.

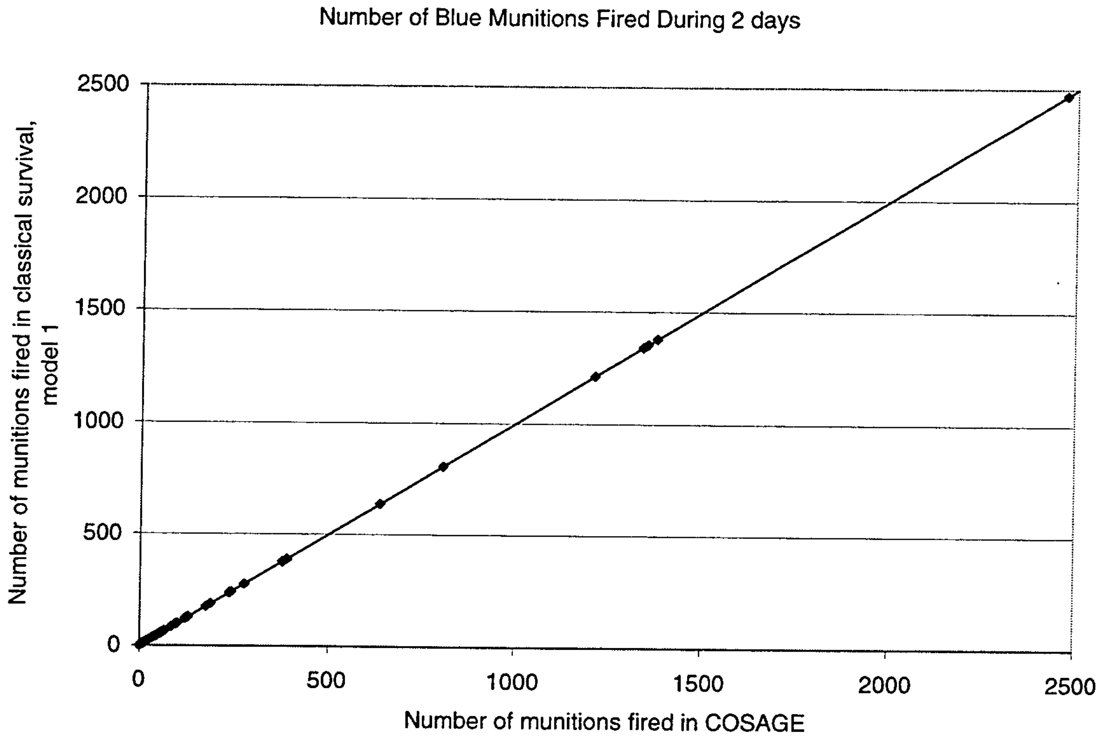
A.1 Classical Survival Model 1 vs. COSAGE Survival, for Blue (No reallocation of fire)



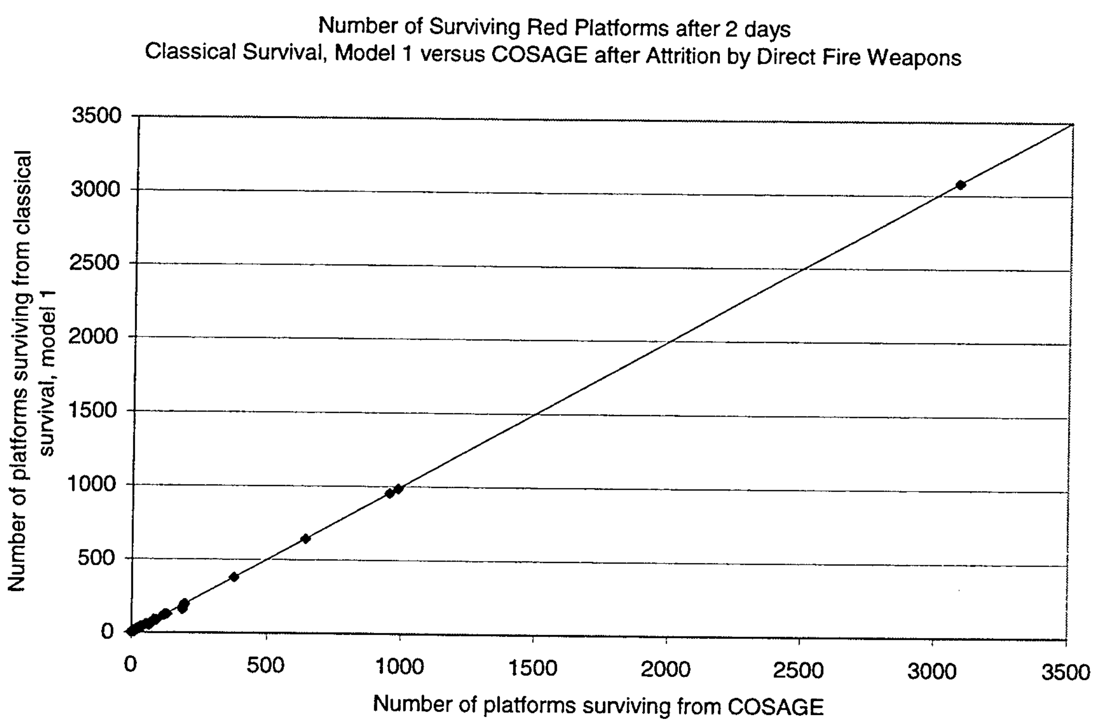
A.2 Classical Survival Model 1 vs. COSAGE Survival, Normalized Difference for Blue (No reallocation)



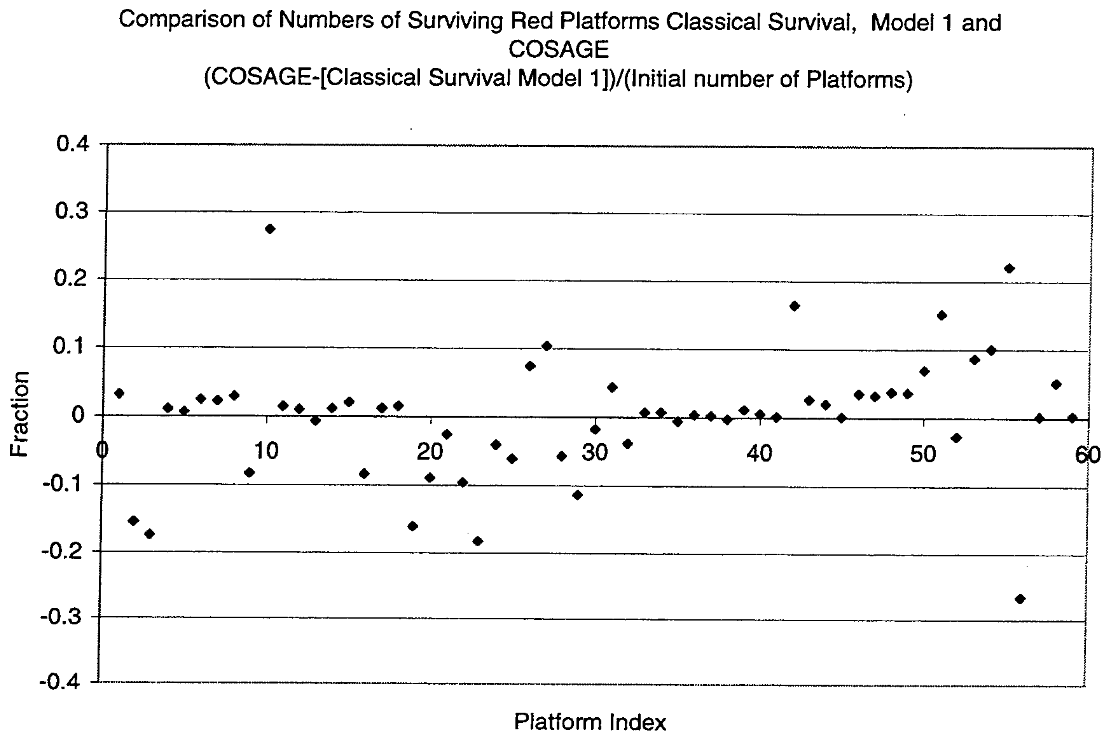
A.3 Classical Survival Model 1 vs. COSAGE Munitions Expenditure: Blue



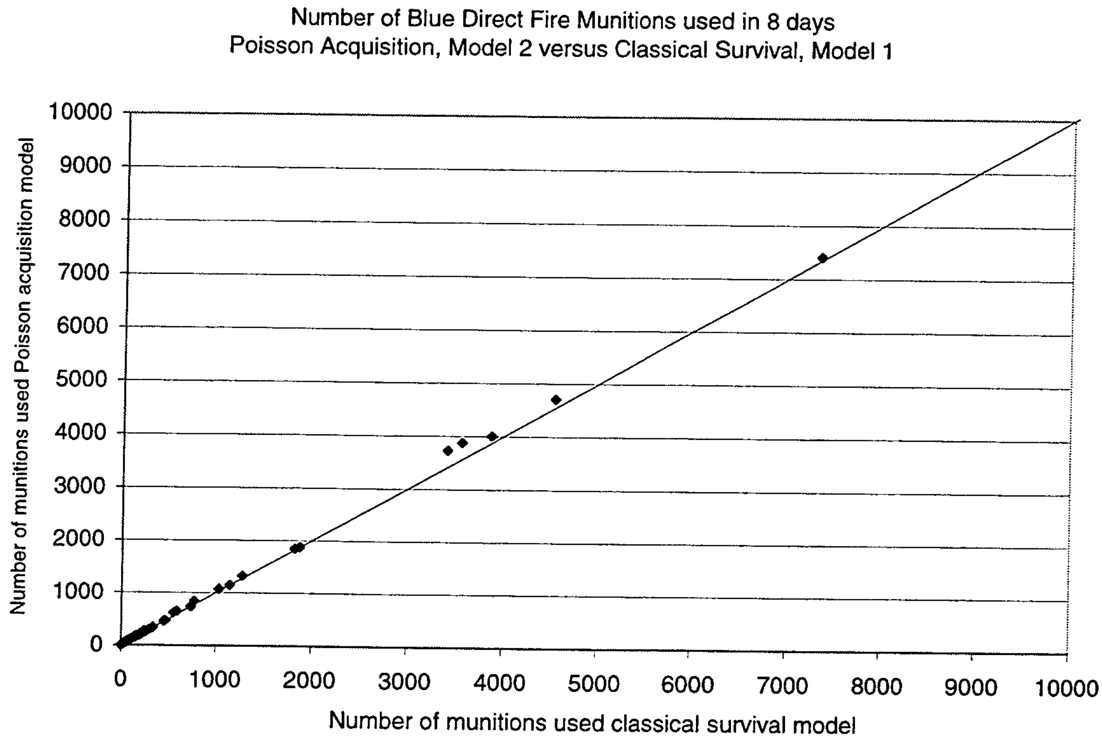
B.1 Classical Survival Model 1 vs. COSAGE Survival, for Red



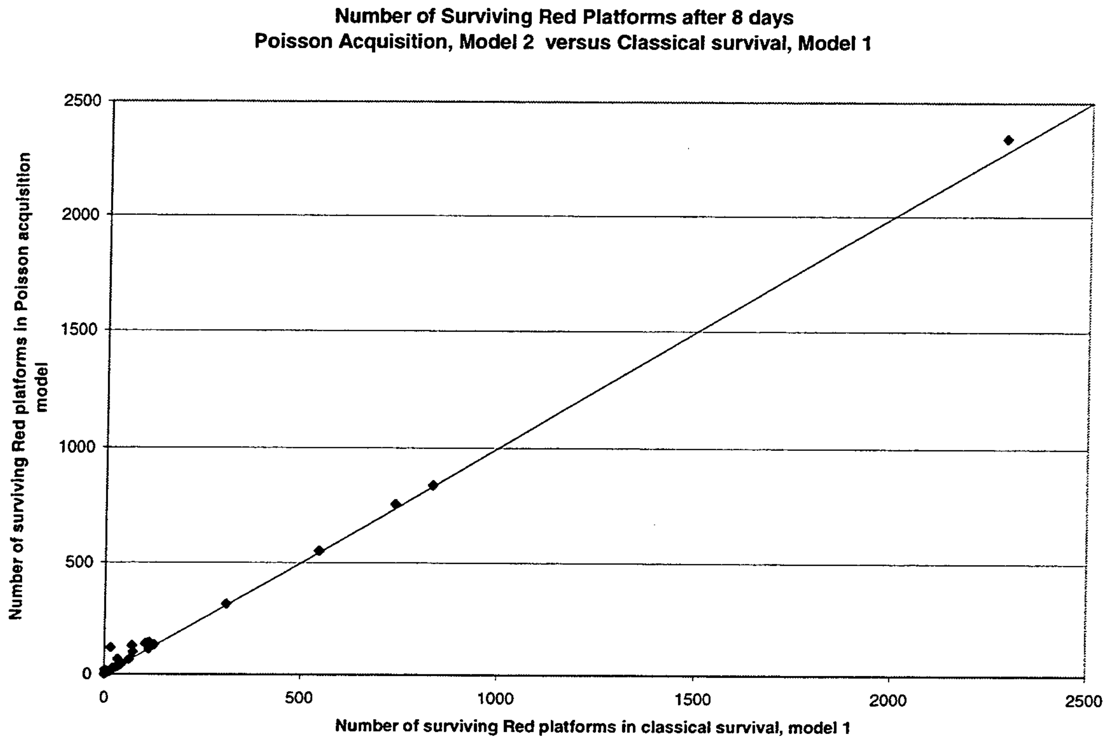
B.2 COSAGE Survival–Classical Survival Model 1: Normalized Difference for Red



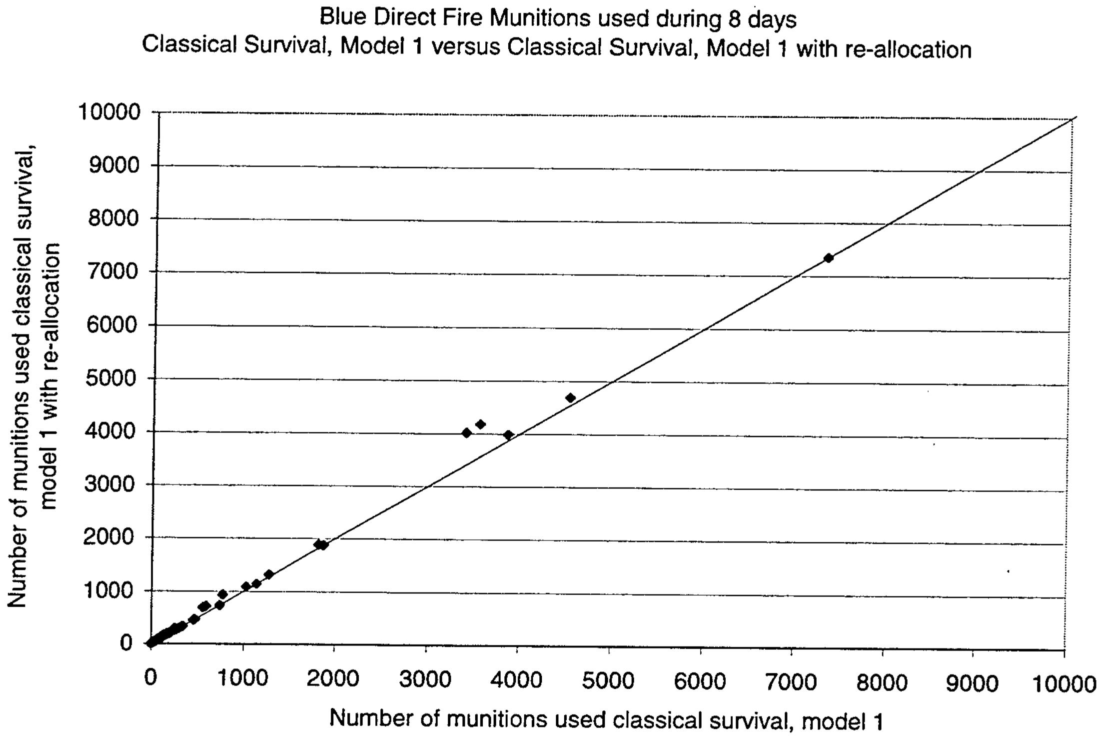
C.1 Munitions Used by Blue in 8 Days: Classical Survival Model 1 vs. Poisson Acquisition, Model 2



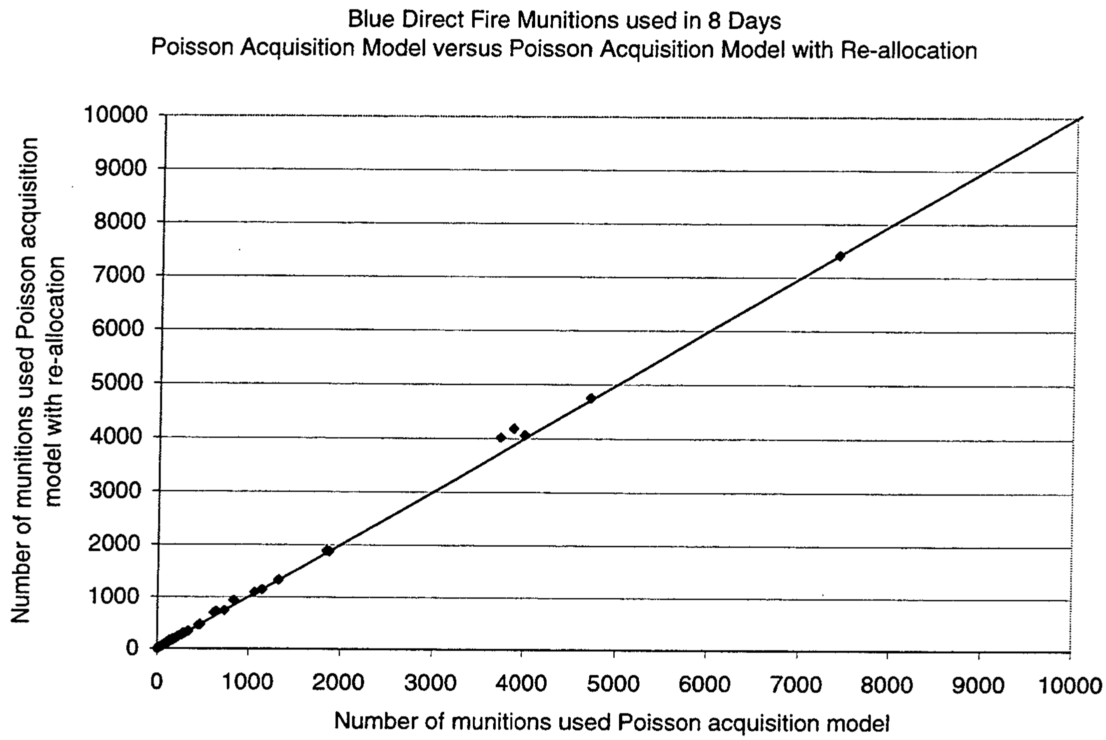
C.2 Red Survivors in 8 Days: Classical Survival Model 1 vs. Poisson Acquisition, Model 2



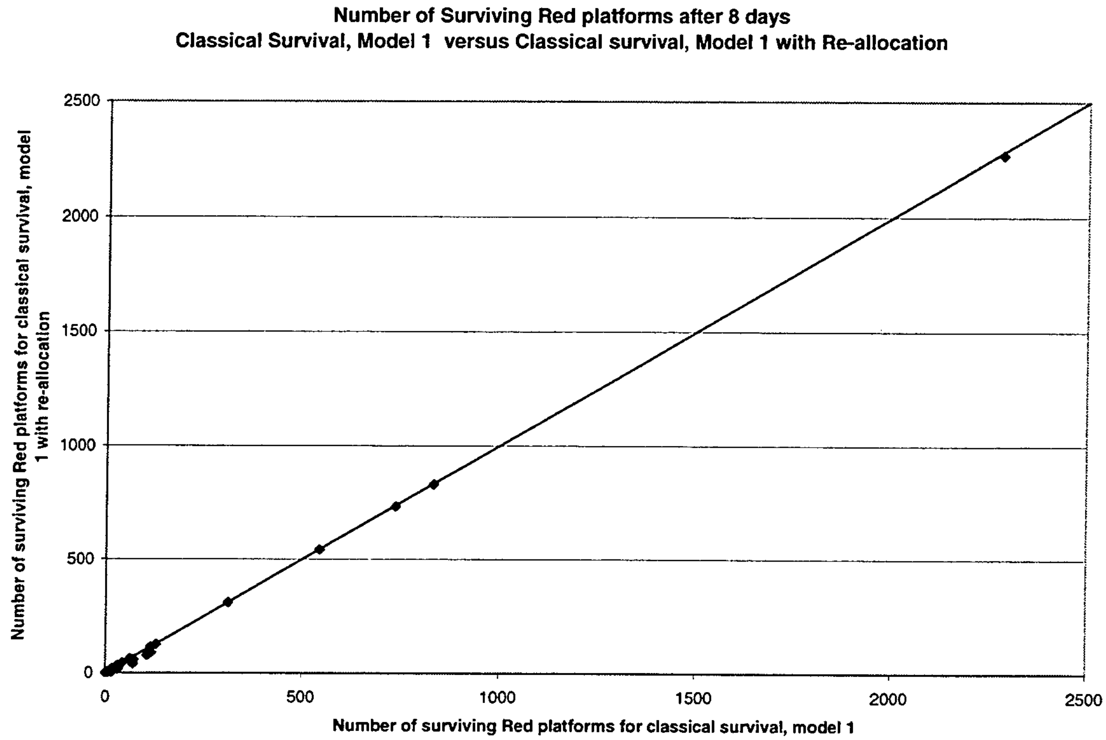
D.1 Munitions Used by Blue in 8 Days: Classical Survival, Model 1, with Proportional Blue Reallocation vs. Classical Survival, Model 1, No Blue Reallocation



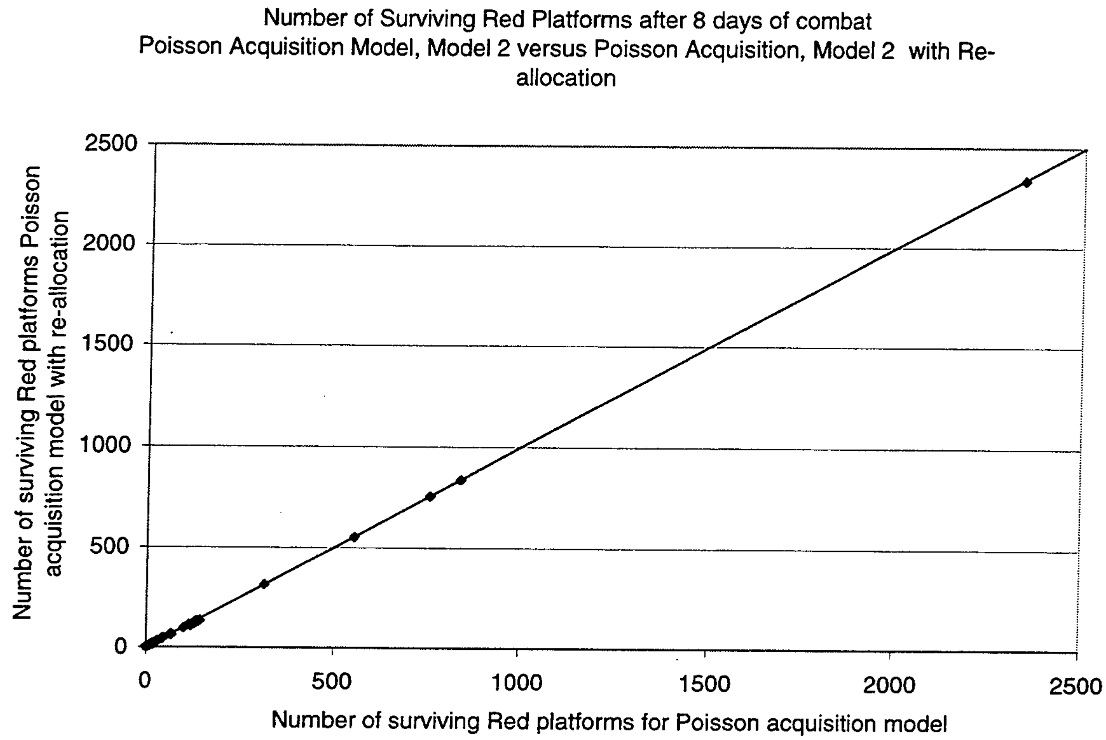
D.2 Munitions Used by Blue in 8 Days: Poisson Acquisition, Model 2 with Proportional Reallocation vs. Poisson Acquisition, Model 2, No Reallocation



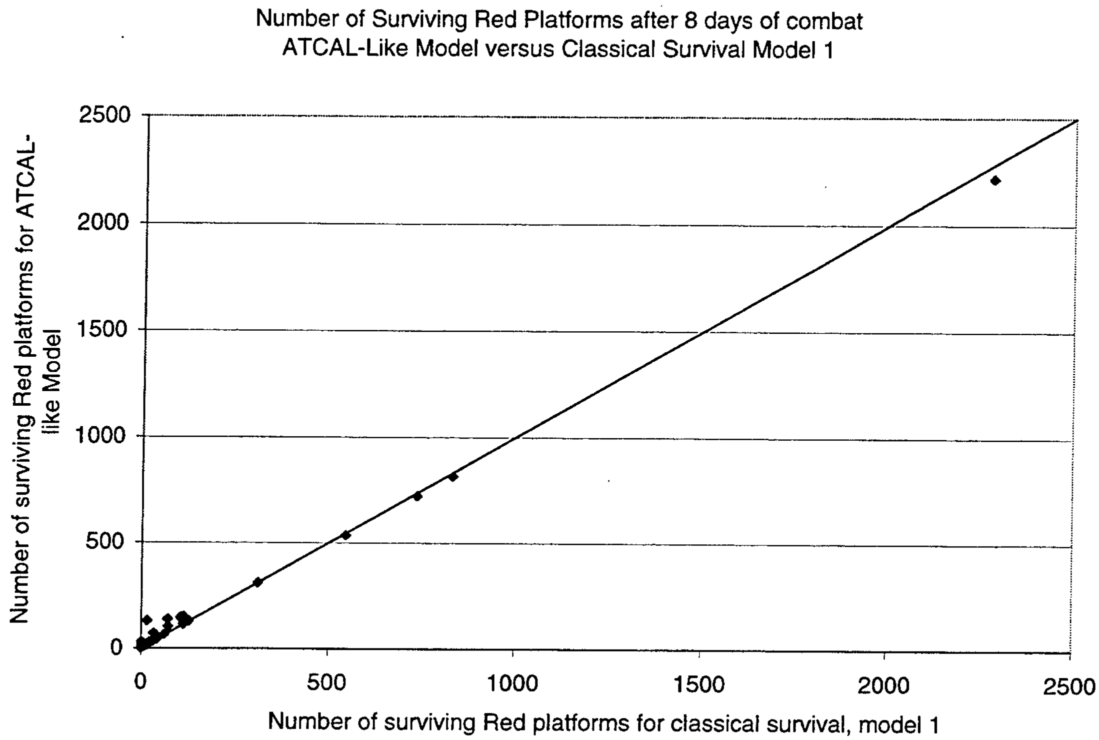
E.1 Red Survival in 8 Days: Classical Survival, Model 1, with Proportional Blue Reallocation vs. Classical Survival, Model 1, No Blue Reallocation



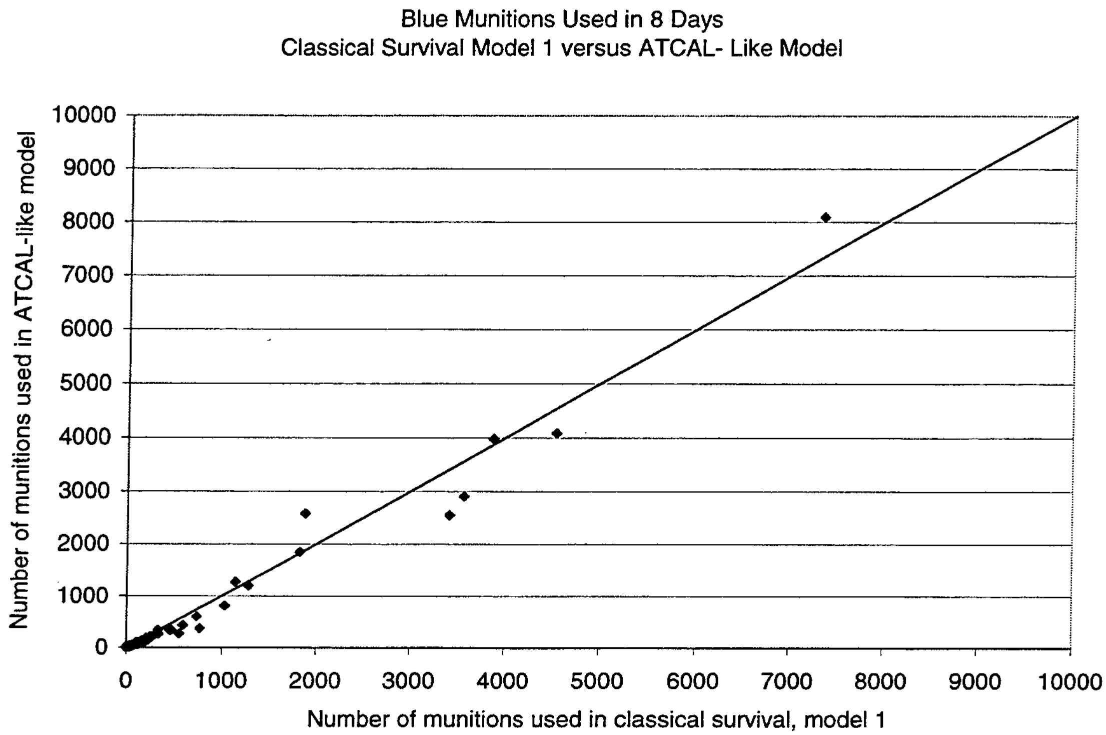
E.2 Red Survival in 8 Days: Poisson Acquisition, Model 2, with Proportional Blue Reallocation vs. Poisson Acquisition, Model 2, No Blue Reallocation



F.1 Red Survival in 8 Days: Classical Survival, Model 1, with ATCAL-like Time-Varying Acquisition Rate vs. Classical Survival, Model 1, No Acquisition Rate Change



F.2 Munitions Used by Blue in 8 Days: Classical Survival, Model 1, with ATCAL-like Time-Varying Acquisition Rate vs. Classical Survival, Model 1, No Acquisition Rate Change



References

1. Agnew, C.E., "Dynamic Modelling and Control of Congestion Prone Systems," *Operations Research*, **24**, 409-419 (1976).

2. Cox, D.R. and D. Oakes, *Analysis of Survival Data*, Chapman and Hall, New York, 1984.

3. Caldwell, W., J. Hartman, S. Parry, A. Washburn, and M. Youngren, "Aggregated Combat Models" Operations Research Department, Naval Postgraduate School, Monterey, CA 2000.

4. Gaver, D.P. and P.A. Jacobs, "Attrition Modeling in the Presence of Decoys: An Operations-Other-Than-War Motivation," *Naval Research Logistics*, **44**, 507-514 (1997).

5. Feller, William, *An Introduction to Probability Theory and Its Applications*, Vol. I, Third Edition, John Wiley & Sons, Inc., New York, 1968.

6. Filipiak, J., *Modeling and Control of Dynamic Flows in Communication Network*, Springer-Verlag, Berlin, 1988.

7. Lowell Bruce Anderson, "Attrition formulas for deterministic models of large-scale combat," *Naval Research Logistics*, **42**, 345-373 (1995).

8. Murray, J.D., *Mathematical Biology*, Springer-Verlag, Berlin, Germany, 1989.

9. Rider, K.L., "A Simple Approximation to the Average Queue With Time-Dependent Queue," *Journal of the ACM*, **23**, 361-367 (1976).

10. Taylor, J.G., *Force-on-Force Attrition Modeling*, Military Applications Section, Operations Research Society of America, Arlington, VA, 1980.

11. Taylor, James G., *Lanchester Models of Warfare*, Military Applications Section, Operations Research Society of America, Arlington, VA, March 1983.

12. US Army Concepts Analysis Agency, COSAGE User's Manual, Vol. 1 – Main report, CAA-D-93-1, April 1993, revised August 1995, Tactical Analysis Division.

13. Washburn, A. "Weapon re-allocation in The Fast Theater Model (FATHM)" (private communication), Naval Postgraduate School, October, 2000.

Appendix A
Stochastic Discrete-Time Models

The previous pages of this report describe a deterministic or "fluid approximation" model, albeit between multi-type forces. Here we show how a related stochastic model can be formulated. Further elaborations are possible, and will be presented subsequently.

The following stochastic state variables describe the situation.

$\boldsymbol{R}(t)$ = *Random variable* denoting number of Reds at beginning (or selected moment) during period t = multiple of $h > 0$: $t = 0, h, 2h, \ldots, 13h, \ldots$ (A.1)

$\boldsymbol{B}(t)$ = Random number of Blues corresponding.

Treat the co-evolution of $\{\boldsymbol{R}(t), \boldsymbol{B}(t), t = 0, h, 2h, \ldots\}$ as a Markov chain.

$$\begin{aligned} \boldsymbol{R}(t+h) &= \boldsymbol{R}_S(t) \\ \boldsymbol{B}(t+h) &= \boldsymbol{B}_S(t) \end{aligned} \tag{A.2}$$

where $\boldsymbol{R}_S(t)$, $\boldsymbol{B}_S(t)$ are the random numbers of survivors of period t that are available for conflict in the subsequent period of duration (h); the state of the system at the beginning of period t is $(\boldsymbol{R}(t), \boldsymbol{B}(t))$, and it evolves to $(\boldsymbol{R}_S(t), \boldsymbol{B}_S(t))$ at the end of that period, which defines the state at the beginning of period $t + h$, and so on.

Poisson Acquisition and Fire

Consider the following ways of assigning Blues to search for and engage Reds.

(A) Sectorization: Direct Fire

Consider Blue-force management first. At the beginning of period t assign $\boldsymbol{B}(t)/\boldsymbol{R}(t)$ distinct Blue shooters, a *group* of Blues, to each Red; e.g. if $\boldsymbol{B}(t) = 200$ and $\boldsymbol{R}(t) = 100$ then the first two Blues are assigned to the first Red, the second two Blues are assigned to the second Red, etc. If fractions arise treat them as fractions of the time interval of duration h.

Let Red forces engaging in sectorized direct (aimed) fire be assigned similarly: $\boldsymbol{R}(t)/\boldsymbol{B}(t)$ distinct Reds per individual Blue. Note that it is not *necessary* that each side be in the same firing mode. The objective of "direct fire" is to assign an equal number of shooters to each target, and not to allow random (e.g. equally likely and independent) target picking. This is analogous to the "aimed or direct fire" protocol that leads to the Lanchester Square Law.

Model the number of contacts/encounters by a Blue group with "its" Red during $(t, t+h)$ as a Poisson process of rate $\lambda_{BR}(\boldsymbol{B}(t)/\boldsymbol{R}(t))$, where the encounter rate parameter λ_B can actually be a function of the period, t, and other variables; some can be decision or control variables. At each encounter with "its" Red let the probability of kill be κ_{BR}, independently of all other events. Therefore the kill process is effectively a *marked terminating Poisson process*, and the probability that a particular Red avoids being killed – survives – period t is $e^{-\lambda_{BR}(\boldsymbol{B}(t)/\boldsymbol{R}(t))}$. We make the convention that killed Reds are removed at the *end* of the period (so they are available to search and shoot during the period.

The same modeling assumption is adopted to describe a single Blue survivorship probability: $e^{-\lambda_{RB}(\boldsymbol{R}(t)/\boldsymbol{B}(t))\kappa_{RB}h}$.

The above allows one to write down the one-step transition probabilities of a Markov chain $\{R(t), B(t), t = 0, h, 2h, \ldots\}$ that is appropriate when the two opponents *both* are in direct-fire mode.

$$P\{\boldsymbol{R}(t+h)=r, \boldsymbol{B}(t+h)=b|\boldsymbol{R}(t), \boldsymbol{B}(t)\} =$$

$$\begin{aligned}
&\binom{\boldsymbol{R}(t)}{r}\left[e^{-\lambda_{BR}(\boldsymbol{B}(t)/\boldsymbol{R}(t))\kappa_{BR}h}\right]^{r}\left[1-e^{-\lambda_{BR}(\boldsymbol{B}(t)/\boldsymbol{R}(t))\kappa_{BR}h}\right]^{R(t)-r} \qquad 0 \le r \le \boldsymbol{R}(t) \\
&\times\binom{\boldsymbol{B}(t)}{b}\left[e^{-\lambda_{RB}(\boldsymbol{R}(t)/\boldsymbol{B}(t))\kappa_{RB}h}\right]^{b}\left[1-e^{-\lambda_{RB}(\boldsymbol{R}(t)/\boldsymbol{B}(t))\kappa_{RB}h}\right]^{B(t)-b} \qquad 0 \le b \le \boldsymbol{B}(t)
\end{aligned} \tag{A.3}$$

It would be possible to refine the above to take account of the order in which Blues and Reds are killed within $(t, t+h)$, but at the cost of greater complexity than is consistent with the present discrete-time model data, e.g. from COSAGE. We make the artificial convention that shot effects (kills) are resolved at the *end* of an interval, so Reds and Blues alive until the end of period t are alive and able to shoot until $(t+h)$-, i.e. the instant before time period $t+h$ starts. This convention could be changed, e.g. to have the settle-up time be the middle of an interval, i.e. at $t+h/2$, but this seems unnecessary and is not done here.

Transition to "Continuous Time": Relation to Lanchester "Square Law"

It is seen from the form of each component of the Markov transition probability (A.3) that the conditional means are

$$E\left[\boldsymbol{R}(t+h)\middle|\boldsymbol{R}(t),\boldsymbol{B}(t)\right]=\boldsymbol{R}(t)e^{-\lambda_{BR}(\boldsymbol{B}(t)/\boldsymbol{R}(t))\kappa_{BR}h} \tag{A.4}$$

and

$$E\left[\boldsymbol{B}(t+h)\middle|\boldsymbol{R}(t),\boldsymbol{B}(t)\right]=\boldsymbol{B}(t)e^{-\lambda_{RB}(\boldsymbol{R}(t)/\boldsymbol{B}(t))\kappa_{RB}h}. \tag{A.5}$$

If $h \to 0$, so time intervals are short, then the two-term Taylor expansion provides

$$E\left[\boldsymbol{R}(t+h)\middle|\boldsymbol{R}(t),\boldsymbol{B}(t)\right]=\boldsymbol{R}(t)\left[1-\lambda_{BR}\left(\boldsymbol{B}(t)/\boldsymbol{R}(t)\right)\kappa_{BR}h+O\left(h^2\right)\right] \tag{A.6}$$

which leads to the differential equation

$$\frac{d}{dt}E\left[\boldsymbol{R}(t)\middle|\boldsymbol{R}(t),\boldsymbol{B}(t)\right]=-\lambda_{BR}\boldsymbol{B}(t)\kappa_{BR}, \tag{A.7}$$

and (formally) removing conditions yields

$$\frac{dE\left[\boldsymbol{R}(t)\right]}{dt}=-\lambda_{BR}E\left[\boldsymbol{B}(t)\right]\kappa_{BR} \tag{A.8}$$

which are precisely the form of the Classical Lanchester Square Law equations. The boundary conditions $E[\boldsymbol{R}(t)] \geq 0$ and $E[\boldsymbol{B}(t)] \geq 0$ must be imposed for correct solution.

Consequently, we argue that the deterministic fluid equations given earlier are a consequence of this stochastic model, and an approximation to aspects of it. Furthermore, variability can be assessed straightforwardly:

$$Var\left[\boldsymbol{R}(t+h)\middle|\boldsymbol{R}(t),\boldsymbol{B}(t)\right]=\left[\boldsymbol{R}(t)e^{-\lambda_{BR}(\boldsymbol{B}(t)/\boldsymbol{R}(t))\kappa_{BR}h}\right]\left[1-e^{-\lambda_{BR}(\boldsymbol{B}(t)/\boldsymbol{R}(t))\kappa_{BR}h}\right] \tag{A.9}$$

and the entirely analogous equation for $Var\left[\boldsymbol{B}(t+h)\middle|\boldsymbol{R}(t),\boldsymbol{B}(t)\right]$.

Firing Options and Immediate (Imperfect) BDA

The model proposed can be *easily implemented* to study several firing options: repeated, information affected shots at the *same* target; they can do so under the guidance of local BDA conducted soon after a shot. Furthermore, the BDA can be rendered as realistically error-afflicted. Incorporation of the various random effects (shot hits and misses, BDA assessments, shot repetitions, repeated BDA, etc.).

For the present we model BDA effects at the unit engagement/shot level. The information presumed used to guide subsequent shots is obtained *at the time of shooting* (just following a shot); it may be provided by the shooter, e.g. from the actual shooting ground unit, from a forward observer, or possibly from an air observer (UAV or helo) — or combination thereof. Note that for what is done here the observers are assumed present and connected to the shooter. This may not be so, but the effect of loss can be modeled in various ways (not included here).

Here are *some* options.

(a) Every engagement involves a single shot, e.g. by Blue or Red. In this familiar case we put $\kappa_{BR} = \kappa_{BR}(1)$ in basic attrition equations (A.3); here $\kappa_{BR}(1)$, or $\kappa_{RB}(1)$, is a single-shot kill probability. It should be regarded as conditional on as many influential variables as are accessible.

(b) Every engagement requires or is ordered to fire (by doctrine) a single *salvo* of s_B shots. Then, simply, assuming independence, the effective engagement kill probability is $\kappa_{BR} = 1 - (1 - \kappa_{BR}(1))^{s_B}$. *Or* there may be a pattern fired; if symmetrically placed, this might be equivalent to $\kappa_{BR} = 1 - (1 - f_1\kappa_{BR}(1))\ (1 - f_2\kappa_{BR}(1))$ for $s_B = 2$, where f_1 (≥ 0) and f_2 (≥ 0) are selected to "automatically" adjust to possible target movement. Again, substitute into (A.3) or the equivalent and solve the equations.

(c) Suppose engagements are quick-time sequences of Shoot, Look, Shoot (SLS) with some termination rule (e.g. second shot is last). The "shot" is then a nearly instantaneous random sequence of individual shots punctuated by brief effect assessments, which are, unfortunately, potentially incorrect:

$c_{BR}(k)$ = P{Blue shot estimated to have killed Red target | Blue shot killed Red target, *or* Red target dead; k means actually killed}

$c_{BR}(m)$ = P{Blue shot estimated to have killed Red target | Blue shot missed Red target, *or* Red target alive; m means actually missed/alive}

Other needed probabilities can be obtained by complementation; let $\bar{c}_{BR}(h)=1-c_{BR}(h)$, $\bar{c}_{BR}(m)=1-c_{BR}(m)$. Now the resultant kill probability of (S,L,S) is

$$\kappa_{BR}=\kappa_{BR}(1)c_{BR}(h)+(1-\kappa_{BR}(1))\bar{c}_{BR}(m)\kappa_{BR}(1);$$

the second shot is here assumed to have the same kill probability; it need not. The expected number of shots of Blue vs. Red per engagement (kill, or not) is, under the doctrine (S,L,S),

$$E[\tilde{\boldsymbol{n}}_{BR}]=1\cdot\kappa_{BR}(1)c_{BR}(h)+2(1-\kappa_{BR}(1)).$$

Another measure of effectiveness is the expected number of *extra* shots taken after a kill, (if by Blue, $\boldsymbol{e}_{BR}$) during/on a given engagement. This is, of course, the consequence of imperfect BDA. In the S,L,S case this is

$$E[e_{BR}]=\bar{c}_{BR}(h)$$

where this is just the probability that the second shot is unnecessary.

(B) Indirect/Area Fire

Suppose Blue conducts Indirect or Area Fire at Red units in a region. Model as follows: a single Blue shooter's "shot" (possibly a pattern or volley) makes a kill on each Red unit independently with probability η_{BR}, so with probability $\bar{\eta}_{BR}\equiv 1-\eta_{BR}$ a particular Red unit survives. It follows that if each Blue shoots according to a Poisson process with rate ξ_{BR} that the particular Red unit survives that one Blue's shot with probability $e^{-\xi_{BR} h \eta_{BR}}$, and survives all of Blue's area fire during $(t, t + h)$ with probability $e^{-\xi_{BR} h \eta_{BR} B(t)}$. Since (by assumption) all Reds are equivalently available, the probability distribution of surviving Reds is binomial

$$P\{\boldsymbol{R}(t+h)=r|\boldsymbol{B}(t),\boldsymbol{R}(t)\}=\binom{\boldsymbol{R}(t)}{r}\left[e^{-\xi_{BR}h\eta_{BR}B(t)}\right]^r\left[1-e^{-\xi_{BR}h\eta_{BR}B(t)}\right]^{R(t)-r}.$$

Under the previous assumptions, e.g. of (A), a similar expression holds for $\boldsymbol{B}(t + h)$. The *conditional* one-step transition probabilities are independent, so we arrive at system transition probability expressions quite analogous to that for (A.3) above. The one-step transition probabilities appropriate if *each side* is exchanging area fire is

$$P\{\boldsymbol{R}(t+h)=r,\boldsymbol{B}(t+h)=b|\boldsymbol{R}(t),\boldsymbol{B}(t)\}=$$

$$\binom{\boldsymbol{R}(t)}{r}\left[e^{-\xi_{BR}h\eta_{BR}\boldsymbol{B}(t)}\right]^r\left[1-e^{-\xi_{BR}h\eta_{BR}\boldsymbol{B}(t)}\right]^{R(t)-r}$$

$$\times\binom{\boldsymbol{B}(t)}{b}\left[e^{-\xi_{RB}h\eta_{RB}\boldsymbol{B}(t)}\right]^b\left[1-e^{-\xi_{RB}h\eta_{RB}\boldsymbol{B}(t)}\right]^{B(t)-b}$$

Note: It is perfectly possible to have one side, Blue, say, attack Red using area fire, while Red attacks Blue using direct fire, or to start with all indirect fire, and transition to direct fire as time increases and range decreases. Combinations of various sorts can be modeled, and the results analyzed.

Appendix B
Variability/Uncertainty Assessment

In actual COSAGE evolutions there are several replications (e.g. $r = 16$) made, and the results eventually averaged to get what we have called the data values $\bar{k}(j_R, j_B;2)$ and $\bar{k}(j_B, j_R;2)$. These can be directly used to *point-estimate* the probability of survival $\bar{\kappa}(x, y; 1)$, and also $R(j_R: t)$, $B(j_B; t)$, $t = 0, h, 2h, \ldots, 17h, \ldots$

In the most accessible documentation the kill data is summarized by mean and (estimated) variance:

$$k(x, y;2) = \frac{1}{r}\sum_{i=1}^{r} k_i(x, y;2) \equiv \bar{k}(x, y;2) \tag{B.1,a}$$

$$\hat{\text{var}}[k(x, y;2)] = \frac{1}{r-1}\sum_{i=1}^{r}\left[k_i(x, y;2) - \bar{k}(x, y;2)\right]^2. \tag{B.1,b}$$

Similar data are available for the number of weapons fired.

A simple computational way of checking for the stability/sampling variability of the results used for predicting (mean) survivors on each side is to (i) re-sample each k-value from a normal or Gaussian distribution $N(\bar{k}, \hat{\text{var}}[k(x, y;2)])$, where r is the number of independent and identical (by assumption) replications (e.g. 16), and use the result(s) to estimate $\hat{\bar{\kappa}}(x, y;2h)$ from (4.10) and (4.18); then (ii) apply this outcome as survival parameters in equations (4.10) and (3.3 a&b); repeat this independently ~ 50 times at $t =$ 4, 8, 10, 12 and examine the results (compute $\bar{R}(j_R;mh)$, $\bar{B}(j_B;lh)$, $m, l = 0, 1, 2, \ldots, 31, \ldots$, and the respective variances and standard deviations). This approach is a way of quantifying the uncertainty in a mean or deterministic approximation. It may be of interest to use as a "certainty equivalent" as part of the process a lower confidence level (roughly $\bar{R}(j_R;t;\bar{q}) - 2\sqrt{\text{var}\,\bar{R}}$, or, possibly better, a value obtained by bootstrapping.

Appendix C
Models for Direct and Indirect Fire

For simplicity assume there is one type of Red target and one type of Blue platform firing one type of Blue weapon. Let $R(0)$ be the initial number of Red targets and $B(0)$ be the initial number of Blue platforms. COSAGE summary data contain the averages of various measures over 16 replications of simulations for 2 days of combat of Red and Blue combatants in various postures.

Aimed (or Direct) Fire Weapons

For aimed (direct fire) fire weapons, COSAGE summary data contain the average number of kills and the average number of shots fired over the $r = 16$ replications. Let $S(0)$, (respectively $\overline{S}(0)$), be the total, (respectively average), number of shots over the 16 replications. $S(0) = 16\overline{S}(0)$. Let $K(0)$, (respectively $\overline{K}(0)$), be the total, (respectively average), number of kills over the 16 replications. Each aimed fire weapon is shot at one Red target. Let $A(j)$ be equal to 1 if the j^{th} shot kills the target and 0 otherwise.

$$K(0) = \sum_{j=1}^{S(0)} A(j)$$

$$E[K(0)] = S(0)\kappa_w(d)$$

Thus an estimate of the probability of kill for an aimed fire weapon is

$$\hat{\kappa}_w(d) = \frac{K(0)}{S(0)} = \frac{\overline{K}(0)}{\overline{S}(0)}$$

Now to avoid having to estimate a kill probability as 0 owing to small sample luck we replace $\hat{\kappa}_w(d) = \frac{K(0)}{S(0)} = \frac{\overline{K}(0)}{\overline{S}(0)}$ by a Bayes estimate that assume kills are binomial with S trials and applies a uniform prior; the result is the estimate of kill $\hat{\kappa}_w(d;b) = \frac{K(0)+1}{S(0)+2} = \frac{\overline{K}(0)+(1/r)}{\overline{S}(0)+(2/r)}$ where r is the number of replications, e.g. 16.

To estimate the probability a particular Red survives all S shots, we assume that the number of shots fired at a particular Red over the $r = 16$ replications is $\frac{S(0)}{rR} = \frac{\overline{S}(0)}{R}$.

Thus, the probability a particular Red survives all the shots is

$$\hat{\kappa}_R(d) = \left[1 - \frac{\overline{K}(0) + (1/r)}{\overline{S}(0) + (2/r)}\right]^{\overline{S}(0)/R}$$

The rate of shooting per Blue over the 48-hour period of COSAGE.

$$\rho(d) = \frac{\overline{S}(0)}{B(0)}$$

An equation to compute the average number of Reds that survive aimed (direct) fire during the time interval $(t, t + h]$ is

$$R(t+h) = R(t)\hat{\kappa}_R(d) = R(t)\left[1 - \frac{\overline{K}(0) + (1/r)}{\overline{S}(0) + (2/r)}\right]^{\rho(d)hB(t)/R(t)}$$

An estimate of the average number of aimed fire weapons shot by one Blue platform against Red targets is $\overline{S}(0)/B(0)$. Assuming an infinite supply of Red targets, an estimate of the expected rate of kill by one Blue platform against Red targets is

$$\rho(d) = (\overline{S}(0)/B(0))\frac{\overline{K}(0) + (1/r)}{\overline{S}(0) + (2/r)}$$

Indirect or Area Fire Weapons

COSAGE summary data record the average number of Red targets that are affected by the indirect fire shots. Let $N_f(0)$ (respectively $\overline{N}_f(0)$) be the total (respectively average) number of Red targets affected by the $S(0)$ (respectively $\overline{S}(0)$) indirect (area fire) (respectively average) fire shots. $N_f(0) = r\overline{N}_f(0) = 16\overline{N}_f(0)$ Let $I_i(j; a; f)$ be equal to 1 if the i^{th} indirect weapon shot affects the j^{th} Red target and be equal to 0 otherwise.

$$N_f = \sum_{i=1}^{S}\sum_{j=1}^{R} I_i(j; a; f)$$

Let X_i be equal to 1 if the i^{th} Red target affected by an indirect weapon shot is killed by the weapon and 0 otherwise. The number of Red targets killed by indirect (area) weapon shots is

$$K(0) = \sum_{i=1}^{N_f(0)} X_i = \sum_{i=1}^{S(0)} \sum_{j=1}^{R(0)} I_i(j;a;k)$$

where $I_i(j; a; k)$ is equal to 1 if the i^{th} indirect (area) weapon shot kills the j^{th} Red target and is 0 otherwise.

$$E[K(0)] = E[\sum_{i=1}^{N_f(0)} X_i] = E[\sum_{i=1}^{S} \sum_{j=1}^{R} I_i(j;a;k)] = S(0)R(0)\kappa_w(a)$$

Thus an estimate of the probability an indirect weapon shot kills a Red target is

$$\hat{\kappa}_w(a) = \frac{K(0)}{S(0)R(0)} = \frac{\overline{K}(0)}{\overline{S}(0)R(0)}$$

where $\overline{K}(0)$ is the average number of Red targets killed over the r COSAGE replications.

To avoid having to estimate a kill probability as 0 owing to small sample luck we replace $\hat{\kappa}_w(a) = \frac{K(0)}{S(0)R(0)} = \frac{\overline{K}(0)}{\overline{S}(0)R(0)}$ by a Bayes estimate that assumes kills are binomial with $S(0)R(0)$ trials and applies a uniform prior; the result is

$$\hat{\kappa}_w(a;b) = \frac{K(0)+1}{S(0)R(0)+2} = \frac{\overline{K}(0)+(1/r)}{\overline{S}(0)R(0)+(2/r)}$$

Estimate the number of shots taken over a single COSAGE replication as $\overline{S}(0)$. The probability a particular Red survives all $\overline{S}(0)$ indirect shots is estimated as

$$\hat{\overline{\kappa}}_R(a) = \left[1 - \frac{\overline{K}(0)+(1/r)}{\overline{S}(0)R(0)+(2/r)}\right]^{\overline{S}}$$

The average number of indirect fire shots fired per Blue over the initial 48-hour period of COSAGE is

$$\rho(a)=\frac{\bar{S}(0)}{B(0)}$$

An equation to compute the average number of Reds surviving all the indirect fire shots in $(t, t+h]$ is

$$R(t+h)=R(t)\hat{\bar{K}}(a)=R(t)\left[1-\frac{\bar{K}(0)+(1/r)}{\bar{S}(0)R(0)+(2/r)}\right]^{\rho(a)hB(t)}$$

The expected number of Reds killed by indirect fire shots during $(t, t+h]$ is

$$D_R(t+h)=R(t)[1-\hat{\bar{\kappa}}_R(a)]=R(t)\left[1-\left[1-\frac{\bar{K}(0)+(1/r)}{\bar{S}(0)R(0)+(2/r)}\right]^{\rho(a)hB(t)}\right]$$

The average number of indirect weapons fired by each Blue at Red targets is $\bar{S}(0)/B(0)$. Assuming an infinite supply of Red targets, an estimated average rate at which one Blue platform using area fire weapons kills Red targets is during the first 48 hour period modeled by COSAGE is

$$\delta(a;0)=\frac{\bar{S}(0)}{B(0)}\frac{\bar{K}(0)+(1/r)}{\bar{S}(0)R(0)+(2/r)}=\rho(a)\frac{\bar{K}(0)+(1/r)}{\bar{S}(0)R(0)+(2/r)}$$

The estimated average rate at which one Blue platform using area fire weapons kills Red targets during $(t, t+h]$ is

$$\delta(a;t)=\rho(a)h\frac{\bar{K}(0)+(1/r)}{\bar{S}(0)R(0)+(2/r)}$$

Example 1:

UH155Z firing area fire munition M483A1 at platform RINTP

Given: Initial number of RINTP = 11068
Initial number of UH155Z = 144
Average total number of indirect shots $\bar{S}_{BR}(j_B, w_B)$ = 26318
Observed number of kills $\bar{K}_{BR}(j_B, w_B, j_R)$ = 509.56
Number of Replications = 16

Bayes probability of kill for a single shot

$$\hat{\kappa}_w(a;b) = \frac{K+1}{SR+2} = \frac{\overline{K}+(1/r)}{\overline{S}R+(2/r)} = \frac{509.56+\frac{1}{16}}{(26318*11068)+(2/16)} = 1.74955E-06$$

The probability a Target survives all shots is

$$q_R(a) = \left[1-\frac{\overline{K}+(1/r)}{\overline{S}R+(2/r)}\right]^{\overline{S}} = [1-1.74955E-06]^{26318} = .954999$$

The estimated average number killed in 48 hours is

$$(\#\text{ targets})\times[1-q_R(a)] = 11068\times(1-.954999) = 498.0682$$

which is close to the COSAGE number of 509.56.

Example 2:

UH155Z firing area fire munition M483A1 at platform ROPR7

Given: Initial number of ROPR7 = 7560
Initial number of UH155Z = 144
Average total number of indirect shots $\overline{S}_{BR}(j_B, w_B)$ = 26318
Observed number of kills $\overline{K}_{BR}(j_B, w_B, j_R)$ = 416.62
Number of Replications = 16

Bayes probability of kill for a single shot

$$\hat{\kappa}_w(a;b) = \frac{K+1}{SR+2} = \frac{\overline{K}+(1/r)}{\overline{S}R+(2/r)} = \frac{416.62+\frac{1}{16}}{(26318*7560)+(2/16)} = 2.09426E-06$$

The probability a Target survives all shots is

$$q_R(a) = \left[1-\frac{\overline{K}+(1/r)}{\overline{S}R+(2/r)}\right]^{\overline{S}} = [1-2.09426E-06]^{26318} = .946375$$

The estimated average number killed in 48 hours is

$$(\#\text{ targets})\times[1-q_R(a)] = 11068\times(1-.946375) = 405.4079$$

which is close to the COSAGE number of 416.62.

Appendix D
ATCAL-Like Firing Rate Model

Versions of ATCAL (abbreviation of *An Attrition Model Using Calibrated Parameters*) Modeling in DISC-O-TIC Style

About twenty years ago, in the early 1980s, the Army, through efforts at the then Combat Analysis Agency (now Center for Army Analysis) or CAA, developed *An Attrition Model Using Calibrated Parameters*, or ATCAL: "a new method for calibrating a set of attrition equations to the results of sample high-resolution simulations." The algorithm developed is a set of attrition equations for "point" (we call it *direct*) and "area" (we call it *indirect*) fire; their algorithm (equations) computes losses by cause (round, or ammunition type), using high-resolution simulation data, as we have done earlier in this paper; thus ATCAL parameters are estimated from high-resolution data, as are ours. ATCAL does recognize target "availability" e.g. visibility, in a simplified probabilistic manner, and also target priorities: such priorities are governed by intrinsic importance to a shooter, but also are higher for those more easy to kill than other available targets. For the present, DISC-O-TIC does not go to such explicit lengths because of limits in COSAGE data available. However, ATCAL does not attempt to model the time-step-evolution or dynamics of combat, whereas DISC-O-TIC does, as seen above. ATCAL seems to assume, rather specifically, that the forces decrease *exponentially* with the duration of a battle, which is an effect that occurs if one reasonably assumes that fire allocation to a target type should decrease as the number of such targets is decreased.

Here is a way of adjusting firing (actually acquisition rate) in DISC-O-TIC to achieve the above effect.

Replace the acquisition rate per shooter in Classical Survival Model 1, $n_{SQ}(\bullet, j_S, w_S; j_Q)$ by a time-dependent rate, $n^{\#}_{SQ}(\bullet, j_S, w_S; j_Q, t)$; here S denotes Shooter type, and Q is quarry/target type; if S is Red then the Q is Blue, and vice versa. Total *shooting rate* per quarry is, after replacement

$$n_{SQ}^{\#}(\bullet, j_S, w_S; j_Q, t)\, S(j_S, w_S, t) / Q(j_Q, t) \tag{D.1}$$

total *killing rate* per quarry is

$$n_{SQ}^{\#}(\bullet, j_S, w_S; j_Q, t)\left(\ln(\overline{\kappa}_{SQ})\right) S(j_S, w_S, t) / Q(j_Q, t). \tag{D.2}$$

Now suppose the shooting rate is actually modified by the factor in square brackets:

$$n_{SQ}^{\#}(\bullet, j_S, w_S; j_Q, t) = \underbrace{\left[\frac{Q(j_Q, t)}{S(j_S, w_S, t)} f_{SQ}(j_S, w_S, j_Q)\right]}_{\substack{\text{correction term that reduces}\\ \text{total Shooting rate if}\\ \text{Quarry/Shooter ratio decreases}}} \underbrace{n_{SQ}(\bullet, j_S, w_S; j_Q)}_{\substack{\text{shooting rate deduced from}\\ (0,2h]\text{ COSAGE data and}\\ \text{Direct Fire Model 1}}} \tag{D.3}$$

$$= n_{SQ}(\bullet, j_S, w_S; j_Q) f_{SQ}(j_S, w_S; j_Q) \frac{Q(j_Q, t)}{S(j_S, w_S, t)};$$

$f_{SQ}(j_S, w_S; j_Q)$ is here a constant to be determined.

This particular form of the Shooter-Quarry rate, if applied across all shooter options leads to geometric decline of the Quarry forces in time although not necessarily at the same rate by type. Prioritization by type can be achieved by adjustment of the tuning constant, f_{SQ}: it can be replaced by $f_{SQ}(j_S, w_S; j_Q)$.

Given the shooting rate per period we get for Model 1 and the ATCAL-like rule,

$$Q(j_Q, t+h) = Q(j_Q, t) \exp\left[\sum_{j_S} \sum_{w_S} n_{SQ}(\bullet, j_S, w_S; j_Q) \ln \overline{\kappa}_{SQ}(j_S, w_S; j_Q) f_{SQ}(j_S, w_S; j_Q)\right]. \tag{D.4}$$

$$\equiv Q(j_Q, t) C_Q(j_Q)$$

so

$$Q(j_Q, t) = Q(j_Q, 0)\left[C_Q(j_Q)\right]^t \tag{D.5}$$

To compute the expenditure of w_S-type weapons/munitions up to the end of time period t it is necessary to sum expenditures during each period: if $M(w_Q;\, t)$ is the expenditure of w_S through period t,

$$\begin{aligned} M(w_Q;t) &= \sum_{x=1}^{t}\sum_{j_S}\sum_{j_Q} n^{\#}_{SQ}(\bullet, j_S, w_S; j_Q, x)\frac{S(j_S,x)}{Q(j_Q,x)}\cdot Q(j_Q,t) \\ &= \sum_{x=1}^{t}\sum_{j_S}\sum_{j_Q} n_{SQ}(\bullet, j_S, w_S; j_Q) f_{SQ}(j_S, w_S; j_Q) Q(j_Q,x) \qquad \text{(D.6)} \\ &= \left[\sum_{j_S}\sum_{j_Q} n_{SQ}(\bullet, j_S, w_S; j_Q) f_{SQ}(j_S, w_S; j_Q)\right]\sum_{x=1}^{t} Q(j_Q,x) \end{aligned}$$

The function $f_{SQ}(j_S,\, w_S;\, j_Q)$ is a potentially arbitrary control function. It follows from (D.5) that *one* way to determine its value is to use the initial COSAGE data; let the COSAGE Shooter-weapon-specific $(j_S,\, w_S)$ number of survivors of Quarry type j_Q be $q(j_S, w_S;\, j_Q,\, 2h)$.

Then, replacing f_{SQ} by its estimate $\tilde{f}_{SQ}$ from available COSAGE data,

$$q(j_S, w_S; j_Q, 2h) = q(j_S, w_S; j_Q, 0)\exp\left[n(\bullet, j_S, w_S; j_Q)\ln\overline{\kappa}_{SQ}\tilde{f}_{SQ}(j_S, w_S; j_Q)\right]. \qquad \text{(D.7)}$$

Now solve for the initial-data-driven value $\tilde{f}_{SQ}$:

$$\tilde{f}_{SQ}(j_S, w_S; j_Q) = \frac{1}{n(\bullet, j_S, w_S; j_Q)\ln\overline{\kappa}_{SQ}(j_S, w_S; j_Q)}\ln\left(\frac{q(j_S, w_S; j_Q, 2h)}{q(j_S, w_S; j_Q, 0)}\right). \qquad \text{(D.8)}$$

In the special case when

(a) $q(j_S, w_S; j_Q, 2h) = q(j_S, w_S; j_Q, 0)$, then we put $f_{SQ}(j_S, w_S; j_Q) = S(j_S, 0)$;

(b) $q(j_S, w_S; j_Q, 2h) = 0$, then we put $f_{SQ}(j_S, w_S; j_Q) = S(j_S, 0)$ for the initial time period and then 0 for the other time periods; and the expenditure of munitions

$$M(j_S, w_S; j_Q, t) = \begin{cases} n_{SQ}(\bullet, j_S, w_S; j_Q) S(j_S, 0) & \textit{for } t = 2h \\ 0 & \text{for } t \geq 3h \end{cases}$$

Discussion. Our basic survival model (D.4) and (D.5) is a consequence of the Shooter being able to perceive his own and the quarry force types and sizes *perfectly* at each time which is manifestly highly optimistic. Sensitivity to this assumption can be studied if only by simulation. This is left for future work.

No particular reason has been given for adjusting acquisition/shooting rate to achieve (nearly precise) geometric/exponential decrease, presumably but not necessarily on both sides. This choice does drive the quarry force size down while limiting munition expenditures, but there are other variations possible in the acquisition/firing rates worth exploration; one or more of these might provide superior results from some standpoint; they should be investigated and clarified in later work.

If more detailed output from COSAGE (or other such high-resolution models) were available, it could be possible to add additional detail to our meta-model, DISCOTIC, that would provide greater insight into choice of suitable acquisition/shooting rates. Also, the cost effectiveness of surveillance, e.g. by helicopter, could be studied by meta-modeling. Such studies could effectively guide future high-resolution model (e.g. COSAGE, but not exclusively) runs, and provide the present models with greater modern detail.

Appendix E

Weapon reallocation

Weapon platforms will fire the same type of munitions at several types of targets. As target platforms are attrited during combat, weapon platforms will re-allocate their fires among the surviving platforms. One perhaps overly simple re-allocation algorithm is to reallocate based on the original proportion of one kind of weapon fired at each type of target; Washburn (2000). In particular, the number of weapons of type w_B fired by platforms of type j_B at targets of type j_R during the time interval (t, t+h] under this re-allocation algorithm is

$$N_{BR}(j_B, w_B, j_R, t+h) - N_{BR}(j_B, w_B, j_R, t) = \frac{1}{h} \frac{N_{BR}(j_B, w_B, j_R, 2h) I(R(j_R, t) > 1)}{\sum_{j_R} N_{BR}(j_B, w_B, j_R, 2h) I(R(j_R, t) > 1)} \quad \text{(E.1)}$$

where $I(R(j_R,t) > 1) = 1$ if $R(j_R,t) > 1$ and 0 otherwise. This reallocation algorithm has the effect of focusing the firing of weapons of this type on the remaining target platforms. It ignores possible delays in target acquisition as the number of target platforms dwindles.

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